BRST Cohomology and Phase Space Reduction in Deformation Quantization

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Abstract

In this article we consider quantum phase space reduction when zero is a regular value of the momentum map. By analogy with the classical case we define the BRST cohomology in the framework of deformation quantization. We compute the quantum BRST cohomology in terms of a 'quantum' Chevalley-Eilenberg cohomology of the Lie algebra on the constraint surface. To prove this result, we construct an explicit chain homotopy, both in the classical and quantum case, which is constructed out of a prolongation of functions on the constraint surface. We have observed the phenomenon that the quantum BRST cohomology cannot always be used for quantum reduction, because generally its zero part is no longer a deformation of the space of all smooth functions on the reduced phase space. But in case the group action is 'sufficiently nice', e.g. proper (which is the case for all compact Lie group actions), it is shown for a strongly invariant star product that the BRST procedure always induces a star product on the reduced phase space in a rather explicit and natural way. Simple examples and counter examples are discussed.

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Contents

1 Introduction

The aim of this article is to give a deformation quantization formulation of the method of BRST (Becchi-Rouet-Stora-Tyutin) cohomology which has been introduced and is frequently used in physics for the quantization of so-called constrained systems (see e.g. [29] and references therein): in symplectic geometry these systems are known as reduced symplectic manifolds.

Deformation quantization has been introduced in [3]. Quantization is formulated as an associative formal deformation, the so-called star product, of the commutative algebra of complex-valued C^∞-functions on a symplectic (or more generally, a Poisson) manifold by a formal series in λ (which corresponds to Plancks constant $\hbar$ in the convergent case) of bidifferential operators such that the term of order zero is pointwise multiplication and the commutator of the first order term is equal to i times the Poisson bracket. By now there are general existence and classification theorems for star products on every Poisson manifold (see [32], and earlier, for symplectic manifolds, [4, 15, 19, 37]). Representation theory for the deformed algebras in the spirit of C^*-algebras has been introduced by [11] by formulating formally positive functionals and formal GNS representations.

The reduction of symplectic manifolds by means of a sufficiently 'nice' Hamiltonian action of a Lie group G (with Lie algebra $\mathfrak{g}$) has been formulated by Marsden and Weinstein (see e.g. [1, Chapter 4]), and the method of BRST cohomology has been transferred to symplectic geometry by Browning and McMullan [12, 36], Kostant and Sternberg [33], and Henneaux and Teitelboim [28], see also [16, 22].

Let us recall briefly the definition of those two objects:

- The starting point of reduction is an Ad^*-equivariant momentum map $J : M \to \mathfrak{g}^*$ for the G-action (see e.g. [1, p. 276]) whose level surface $C := J^{-1}(\{0\})$ plays the rôle of the constraint surface in physics. The reduced phase space M_{red} is then the symplectic manifold C/G where usually the action of G on C is supposed to be proper and free in order to guarantee a compatible differentiable structure on M_{red}. In that case thanks to the Dirac method we can see the space $C^\infty(M_{\mathrm{red}})$ as a quotient $\mathcal{B}_C/\mathcal{I}_C$ where $\mathcal{I}_C$ is the vanishing ideal of C and $\mathcal{B}_C$ is its normalizer with respect to the Poisson bracket.

- For the BRST method the Poisson algebra $C^\infty(M)$ is tensored by the Graßmann algebra over the direct sum of the Lie algebra and its dual (the latter space $\bigwedge\mathfrak{g}^* \otimes_{\mathbb{R}} \bigwedge\mathfrak{g}$ itself becomes a super Poisson algebra by means of the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$) to an extended super Poisson algebra $\mathcal{A}$ which is called the classical BRST algebra. Roughly speaking, the super Poisson bracket of a suitable linear combination of the Lie structure and the momentum map, Θ, serves as an odd Hamiltonian super-derivation of square zero, the so-called classical BRST operator, in the extended super Poisson algebra $\mathcal{A}$. There is in addition the so-called ghost number or total degree derivation on $\mathcal{A}$ which is equal to $k - l$ on each subspace $\bigwedge^k\mathfrak{g}^* \wedge \bigwedge^l\mathfrak{g} \otimes C^\infty(M)$. It turns out that the classical BRST operator is the total differential of a double complex whose vertical differential is a Chevalley-Eilenberg differential whereas the horizontal differential is twice the classical Koszul boundary operator.

  BRST cohomology gives then a very nice method to describe the space $C^\infty(M_{\mathrm{red}})$, a method that we shall use for quantization. In case 0 is a regular value of the momentum map (and the action of G on M is nice: for example proper and free) the 'ghost number zero part' of the cohomology of the classical BRST operator is known to be isomorphic (as a Poisson algebra) to the space $C^\infty(M_{\mathrm{red}})$.

Recently there have been several articles in which phase space reduction has been dealt with in deformation quantization: Fedosov has formulated symplectic reduction in his scheme of star

products for $U(1)$-actions [18] and for general compact groups [21]. In particular situations, phase space reduction methods have been used to compute explicit formulas for star products on all complex projective spaces [6, 7], on Graßmann manifolds [43, 44], and for one-dimensional Lie algebras [24, 25]. The method of BRST cohomology has been successfully formulated for geometric quantization (see e.g. [31] and [45], and references therein), but there does not yet seem to be a treatment of BRST cohomology in deformation quantization although BRST cohomology "seems to be well-suited to the recent work on the deformation approach to quantization", see [36, p. 428].

In this article, we shall give a quantum version of the BRST method described before to get similar results for deformation quantization on constrained systems. But we shall not restrict our study only to the nice cases (such as regular value of the momentum map, proper and free group action). Actually, we are convinced that a treatment of BRST cohomology in deformation quantization has several advantages:

Firstly, physicists using BRST cohomology methods often complain about operator ordering problems which forces them to *a priori assume the existence* of a quantum BRST (cohomology) operator: to quote Henneaux and Teitelboim [29, p. 297]: "It will be assumed that one can find a charge Ω satisfying the nilpotence and hermiticity conditions [...] Unlike the situation in the classical case, there is no *a priori* guarantee that this can always be done starting from a classical theory for which $[\Omega, \Omega] = 0$, since the ordering of the factors comes in crucially." In contrast to that, deformation quantization can also be viewed as a theory consistently overcoming and even classifying *all* possible operator orderings in situations where differential operator representations of the deformed algebra (for example in a symbol calculus on cotangent bundles, see e.g. [8–10, 42] for a treatment on curved configuration spaces) are possible. Moreover, there are general theorems in deformation quantization about the quantization of proper Hamiltonian group actions [20, p. 180–183].

Secondly, in deformation quantization (as in the C^*-algebra theory) the observable algebra is the principal object whereas representations are subordinate. Therefore it is rather natural to check whether a classical BRST operator simply remains a cohomology operator when the super Poisson bracket is replaced by the super-commutator of a $\mathbb{Z}_2$-graded star product.

Thirdly, by its very definition deformation quantization allows us to control the classical limit after the quantum reduction which often ends up (in other quantization schemes) with abstract quotient algebras.

In this article we have come to the following principal results:

i.) For every star product $*$ on M covariant under the group action (which can be achieved for every proper Lie group action), i.e.

$$\langle J, \xi\rangle * \langle J, \eta\rangle - \langle J, \eta\rangle * \langle J, \xi\rangle = \mathrm{i}\lambda\langle J, [\xi, \eta]\rangle \qquad \forall \xi, \eta \in \mathfrak{g},$$

where $\langle\cdot,\cdot\rangle$ denotes the natural pairing, we construct a one-parameter family of formal associative deformations of the classical extended super Poisson algebra, $(\mathcal{A}[[\lambda]], \star_\kappa)_{\kappa\in[0,1]}$, which are all equivalent by an explicit equivalence transformation S_κ and for which the corresponding BRST charge Θ_κ has square zero (Section 5).

ii.) We compute the family of quantum BRST operators $\boldsymbol{\mathcal{D}}_\kappa$ in $(\mathcal{A}[[\lambda]], \star_\kappa)$, i.e. the graded star product commutators with Θ_κ. It turns out that the quantum BRST operator which seems to be the most 'natural' from the point of view of Clifford algebras (which we called the Weyl-ordered BRST operator $\boldsymbol{\mathcal{D}}_{\mathrm{W}}$, and which corresponds to $\kappa = 1/2$) looks rather complicated and not very encouraging concerning cohomology computations. But luckily the quantum BRST operator corresponding to $\kappa = 0$ (which we called standard ordered BRST operator

$\boldsymbol{\mathcal{D}}_{\mathrm{S}}$) —which is conjugate by $S_{1/2}$ to $\boldsymbol{\mathcal{D}}_{\mathrm{W}}$— again defines a double complex (Section 5) thus giving rise to deformed versions of the classical Koszul boundary operator and the classical Chevalley-Eilenberg differential.

iii.) For every regular level-zero surface C we compute the quantum BRST cohomology of $(\mathcal{A}[[\lambda]], \boldsymbol{\mathcal{D}}_{\mathrm{S}})$ (which is again a $\mathbb{Z}$-graded associative algebra) by means of a *deformed augmentation*, i.e. a quantised version of the linear map restricting functions on the manifold to C: the result is that the quantum BRST cohomology is isomorphic in an explicit way (using deformed versions of classical chain homotopies of the classical Koszul complex) to the Chevalley-Eilenberg cohomology of the Lie algebra $\mathfrak{g}$ with $\mathfrak{g}$-module $C^\infty(C)[[\lambda]]$ where the representation is a deformation of the usual Lie derivative of the vector fields of the classical $\mathfrak{g}$-action. Moreover the quantum BRST cohomologies of the above quantum BRST operators $\boldsymbol{\mathcal{D}}_\kappa$ are all isomorphic as associative algebras to the cohomology of the standard ordered operator. Finally, we also arrive at the isomorphy of the quantum BRST cohomology algebra and a Dirac-type picture: we define a deformed version of the vanishing ideal $\mathcal{I}_C$ of the constraint surface, a certain left ideal $\boldsymbol{\mathcal{I}}_C$ of $(C^\infty(M)[[\lambda]], *)$, and its idealiser $\boldsymbol{\mathcal{B}}_C$ modulo $\boldsymbol{\mathcal{I}}_C$ turns out to be naturally isomorphic to the quantum BRST algebra (Section 6).

iv.) The natural question arising in view of the preceding result is the following: even in the nice case (regularity, proper and free action), does this deformed Chevalley-Eilenberg cohomology for ghost number zero reflect in an isomorphic manner the space of functions on the reduced phase space (in case this space exists)? As we shall show in Section 7 by a simple example dealing with a Hamiltonian circle action the answer is in general *no*! It may happen that momentum map and star product are so 'ill-adjusted' that the zeroth quantised Chevalley-Eilenberg cohomology of $\mathfrak{g}$ on the constraint surface C becomes 'much smaller' than the classical cohomology which is in bijection with $C^\infty(M_{\mathrm{red}})$.

v.) However, as we show in Section 8 there are large classes of examples in which the aforementioned pathology does not occur: the first class is the family of proper Hamiltonian G-spaces for which there always exist strongly invariant star products and G-equivariant chain homotopies and prolongations. Here, using such a strongly invariant star product, classical and quantum Chevalley-Eilenberg cohomologies of the Lie algebra $\mathfrak{g}$ on the constraint surface are simply equal. Moreover we get fairly explicit formulas for the star product on the reduced phase space in terms of the star product on the original symplectic manifold, an equivariant prolongation map and a deformed restriction map. This formula is particularly simple for global G-invariant functions thus serving to quantize integrable systems obtained by reduction. The second class of examples consists of those situations where the first classical Chevalley-Eilenberg cohomology group of the Lie algebra $\mathfrak{g}$ on the constraint surface is zero. For particular cases this is satisfied when the first de Rham cohomology of the Lie group vanishes.

The paper is organised as follows: in the first three Sections 2, 3, and 4 we recall basic concepts and results in deformation quantization, geometry and Koszul complex for constraint surfaces, and classical BRST theory, respectively. As stated above, the main results of this paper are contained in Section 5, Section 6, Section 7, and Section 8. In Section 9 we give a short conclusion and discuss further problems and questions arising with our approach.

Notation: Tensor products $\otimes$ are usually taken over $\mathbb{C}$. Otherwise the ring will be indicated as subscript. Moreover, $C^\infty(M)$ always denotes the space of smooth complex-valued functions on M.

Finally, $H^\bullet$ indicates a $\mathbb{Z}$-grading of a module H and analogously $\mathcal{A}^{\bullet,\bullet}$ denotes a $\mathbb{Z}\times\mathbb{Z}$ grading. A homogeneous map Φ of degree k is denoted by $\Phi : H^\bullet \to H^{\bullet+k}$.

2 Star products and Hamiltonian Lie group/algebra actions

In this section we shall recall some basic concepts of deformation quantization and Hamiltonian Lie group and Lie algebra actions in order to establish our notation, see also e.g. [1].

We consider a Poisson manifold (M,Λ), i.e. a smooth manifold M with a Poisson tensor field $\Lambda \in \Gamma^\infty(\bigwedge^2 TM)$ such that the Schouten bracket $[\Lambda,\Lambda]$ vanishes, see e.g. [13]. Then $\{f,g\} := \Lambda(df,dg)$ defines a Poisson bracket which turns $C^\infty(M)$ into a Poisson algebra. Here we always consider *complex-valued* functions and tensor fields. The vector field $X_f := \Lambda(df,\cdot)$ is called the Hamiltonian vector field of $f \in C^\infty(M)$. A particular case of a Poisson manifold is a symplectic manifold (M,ω) where the symplectic form $\omega \in \Gamma^\infty(\bigwedge^2 T^*M)$ is a closed, non-degenerate two-form. In this case the Hamiltonian vector field of f is defined by $i_{X_f}\omega = df$ and the Poisson bracket is $\{f,g\} = \omega(X_f, X_g)$ whence the Poisson tensor Λ is just the 'inverse' of $-\omega$.

Now we consider the space $C^\infty(M)[[\lambda]]$ of formal power series in a formal parameter λ as $\mathbb{C}[[\lambda]]$-module. Then a *star product* $*$ for (M,Λ) is a $\mathbb{C}[[\lambda]]$-bilinear, associative deformation of the pointwise product of $C^\infty(M)$ such that

$$f * g = \sum_{r=0}^\infty \lambda^r C_r(f,g), \tag{1}$$

where $C_0(f,g) = fg$, $C_1(f,g) - C_1(g,f) = \mathrm{i}\{f,g\}$, and all C_r are bidifferential operators vanishing on constants whence $1 * f = f = f * 1$, see [3]. Sometimes further requirements are made by specifying certain parity or reality condition for the C_r. Furthermore a star product is of *Vey type* if the bidifferential operator C_r is of order r in each argument for all r. One might also take *local* operators C_r instead of bidifferential ones but we shall deal only with bidifferential ones for simplicity. The formal parameter λ plays the rôle of Planck's constant $\hbar$ and may be substituted by $\hbar$ in convergent situations. The existence of such star products was shown in the symplectic case by DeWilde and Lecomte [15], Fedosov [17, 19], and Omori, Maeda, and Yoshioka [39], and recently by Kontsevich in the general Poisson case [32]. Two star products $*$ and $*'$ are called *equivalent* if there exists a formal series of differential operators $T = \mathsf{id} + \sum_{r=1}^\infty \lambda^r T_r$ such that $T(f * g) = Tf *' Tg$ for all $f,g \in C^\infty(M)[[\lambda]]$. The classification up to equivalence was done by Nest and Tsygan [37, 38], Deligne [14], Bertelson, Cahen and Gutt [4], and Kontsevich [32].

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra with dual space $\mathfrak{g}^*$. Recall that a *Lie algebra action* of $\mathfrak{g}$ on M is a linear anti-homomorphism $\xi \mapsto \xi_M$ of $\mathfrak{g}$ into the Lie algebra of all vector fields on M. It follows that these vector fields define a representation ϱ_M of $\mathfrak{g}$ in the space $C^\infty(M)$ by

$$\varrho_M(\xi)(f) := -\xi_M(f). \tag{2}$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Any (left) *Lie group action* of $\Phi : G\times M \to M : (g,x) \mapsto \Phi_g(x)$ defines a Lie algebra action by means of its infinitesimal generators $\xi_M := \frac{d}{dt}\Phi_{\exp(t\xi)}\big|_{t=0}$. Recall that a Lie algebra action (or a Lie group action) on a Poisson manifold (M,Λ) is called Hamiltonian (see e.g. [1, Sect. 4] for details) if and only if there is a *momentum map* of the action, i.e. a C^∞-map $J : M \to \mathfrak{g}^*$ such that for every $\xi \in \mathfrak{g}$

$$X_{\langle J,\xi\rangle} = \xi_M. \tag{3}$$

Moreover we require equivariance of J with respect to the coadjoint representation of $\mathfrak{g}$ and G, i.e. $\{\langle J, \xi\rangle, \langle J, \eta\rangle\} = \langle J, [\xi, \eta]\rangle$ for all $\xi, \eta \in \mathfrak{g}$ in the case of a Hamiltonian Lie algebra action and $J(\Phi_g(x)) = \mathrm{Ad}^*(g)J(x)$ for all $g \in G$ and $x \in M$ in case of a Hamiltonian Lie group action. In the last case the quadruple (M, Λ, G, J) is usually called a *Hamiltonian G-space*. We shall speak of a *Hamiltonian $\mathfrak{g}$-space* $(M, \Lambda, \mathfrak{g}, J)$ in the more general case of a Hamiltonian Lie algebra action. In the symplectic case we shall denote this by $(M, \omega, \mathfrak{g}, J)$ and (M, ω, G, J), respectively.

For a physically reasonable quantization procedure one certainly has to impose more conditions on a star product beside the defining ones, since particular properties of the Poisson manifold, as e.g. symmetries, should be preserved under quantization. This leads to the various definitions of 'invariance' for star products under a given classical Lie group or Lie algebra action: In the context of deformation quantization the following notions of invariance are commonly used, see e.g. [2]: The star product $*$ is called

- *invariant* if $\Phi_g^*(f*h) = \Phi_g^*(f)*\Phi_g^*(h)$ for a Hamiltonian Lie group action and, more generally, for a Hamiltonian Lie algebra action $\{\langle J, \xi\rangle, f * h\} = \{\langle J, \xi\rangle, f\} * h + f * \{\langle J, \xi\rangle, h\}$ for all $g \in G$ resp. $\xi \in \mathfrak{g}$ and $f, h \in C^\infty(M)[[\lambda]]$,
- *covariant* if $\langle J, \xi\rangle * \langle J, \eta\rangle - \langle J, \eta\rangle * \langle J, \xi\rangle = \mathrm{i}\lambda\langle J, [\xi, \eta]\rangle$ for all $\xi, \eta \in \mathfrak{g}$ for both types of Hamiltonian action, and finally,
- *strongly invariant* if $\langle J, \xi\rangle * f - f * \langle J, \xi\rangle = \mathrm{i}\lambda\{\langle J, \xi\rangle, f\}$ for all $\xi \in \mathfrak{g}$ and $f \in C^\infty(M)[[\lambda]]$ for both types of Hamiltonian action.

Then clearly strong invariance implies both invariance and covariance in case the Lie group G is connected. Furthermore one can allow quantum corrections to the momentum map leading to the notion of a *quantum momentum map*, see e.g. [6, 47]. We consider a formal series $\boldsymbol{J} = \sum_{r=0}^\infty \lambda^r \boldsymbol{J}_r : M \to \mathfrak{g}^*[[\lambda]]$ of smooth functions $\boldsymbol{J}_r : M \to \mathfrak{g}^*$ such that $\boldsymbol{J}_0 = J$ is the classical momentum map and $\boldsymbol{J}$ satisfies $\langle \boldsymbol{J}, \xi\rangle * \langle \boldsymbol{J}, \eta\rangle - \langle \boldsymbol{J}, \eta\rangle * \langle \boldsymbol{J}, \xi\rangle = \mathrm{i}\lambda\langle \boldsymbol{J}, [\xi, \eta]\rangle$ for all $\xi, \eta \in \mathfrak{g}$. In this case the star product will be called *quantum covariant*, and clearly covariance with respect to J implies quantum covariance for $\boldsymbol{J} = J$. Moreover quantum covariance implies that the Lie algebra $\mathfrak{g}$ acts by 'inner' derivations on the algebra $(C^\infty(M)[[\lambda]], *)$, where the representation $\boldsymbol{\varrho}_M$ is given by

$$\boldsymbol{\varrho}_M(\xi) = \frac{1}{\mathrm{i}\lambda}\mathrm{ad}_*(\langle \boldsymbol{J}, \xi\rangle) \tag{4}$$

for $\xi \in \mathfrak{g}$. Here ad_* stands for taking commutators with respect to the star product $*$. This motivates the following definition:

Definition 1 *The quadruple $(M, *, \mathfrak{g}, \boldsymbol{J})$ is called Hamiltonian quantum $\mathfrak{g}$-space if $(M, *)$ is a Poisson manifold with star product such that $\boldsymbol{J}$ is a quantum momentum map and $*$ is quantum covariant under $\mathfrak{g}$.*

Finally a quantum $\mathfrak{g}$-space $(M, *, \mathfrak{g}, \boldsymbol{\varrho}_M)$ is defined to be a Poisson manifold with star product such that the Lie algebra $\mathfrak{g}$ acts via $\boldsymbol{\varrho}_M$ by not necessarily inner star product derivations on $C^\infty(M)[[\lambda]]$. Given a Hamiltonian quantum $\mathfrak{g}$-space $(M, *, \mathfrak{g}, \boldsymbol{J})$ we call $(M, \Lambda, \mathfrak{g}, J = \boldsymbol{J}_0)$ the corresponding *classical limit*.

We shall now consider particular Hamiltonian group actions which all imply the existence of a strongly invariant star product in the symplectic case. Recall that a Lie group action $\Phi : G \times M \to M$ is called *proper* if the map $\hat{\Phi} : G \times M \to M \times M : (g, m) \mapsto (\Phi(g, m), m)$ is proper, i.e. inverse images of compact sets are compact. Equivalently, for any sequences $(x_n)_{n\in\mathbb{N}}$ in M and $(g_n)_{n\in\mathbb{N}}$ in G such that $x_n \to x$ and $\Phi(g_n, x_n) \to y$ there is a subsequence of $(g_n)_{n\in\mathbb{N}}$ converging to $g \in G$ such

that $y = \Phi(g, x)$. Proper group actions always have closed orbits and compact isotropy groups (see also [1, Chap. 4]). One has the following implications in case (M, ω) is symplectic:

$$\begin{matrix} \text{compact} \\ \text{group} \\ \text{action} \end{matrix} \Longrightarrow \begin{matrix} \text{proper} \\ \text{group} \\ \text{action} \end{matrix} \Longrightarrow \begin{matrix} \text{connection} \\ \text{preserving group} \\ \text{action} \end{matrix} \Longrightarrow \begin{matrix} \text{existence of} \\ \text{strongly invariant} \\ \text{star product} \end{matrix} \qquad (5)$$

While the first two implications are well-known general geometrical results on Lie group actions, see e.g. [40, Thm. 4.3.1], the existence of a strongly invariant star product in the case where the group leaves invariant a connection is proved by Fedosov's techniques [20, Sect. 5.8].

3 Geometry of constraint surfaces and classical phase space reduction

In this section we shall briefly recall the relation between the geometry of constraint surfaces in a (Poisson) manifold M and certain subspaces of smooth functions on M.

Let C be a regular closed sub-manifold of a manifold M and denote by $\iota : C \to M$ the canonical embedding. Let $\mathcal{I}_C$ denote the *vanishing ideal of* C, i.e. the subspace of $C^\infty(M)$ of all those functions which vanish on C. The following well-known lemma shows the existence of a *prolongation* of smooth complex-valued functions on C in an open neighbourhood of C:

Lemma 2 *Let C be a regular closed sub-manifold of a manifold M. Then there is an open neighbourhood U of C and a subspace $\mathcal{F}_C$ of $C^\infty(M)$ such that:*

i.) Each $f \in \mathcal{F}_C$ is supported in U.

ii.) The restriction $\iota^* : \mathcal{F}_C \to C^\infty(C)$ is a bijection. We shall call its inverse prol *the prolongation of $\phi \in C^\infty(C)$ to M.*

iii.) The space $C^\infty(M)$ decomposes into the direct sum $\mathcal{F}_C \oplus \mathcal{I}_C$.

iv.) Let $\phi \in C^\infty(C)$ have compact support. Then $\mathrm{prol}\,\phi$ *has compact support in M.*

In particular this entails that the map ι^* induces a canonical bijection of the quotient $C^\infty(M)/\mathcal{I}_C$ onto $C^\infty(C)$.

Note that we have borrowed the notation prol from a paper by Glößner [24, 25], but Glößner denotes the projection $\mathrm{prol}\,\iota^*$ by prol.

Suppose next that there is a proper action of a Lie group G on M such that the sub-manifold C is invariant under this action. Then we have the following

Lemma 3 *Suppose that a Lie group G properly acts on the manifold M such that the sub-manifold C is preserved under this action. Then the open neighbourhood U and the subspace $\mathcal{F}_C$ satisfying the properties of the preceding Lemma can in addition be chosen to be invariant under this action such that the prolongation intertwines the action on C with the action on M.*

PROOF: The existence of a G-invariant prolongation map is shown by using a *G-invariant tubular neighbourhood* U of C in M: consider the conormal bundle $E := \{\alpha \in T^*M|_C \mid \alpha(v) = 0\ \forall v \in T_{\tau(\alpha)}C\}$ (where τ denotes the cotangent bundle projection) on which G acts in a canonical manner such that τ is G-equivariant; a G-invariant tubular neighbourhood consists in the following data: a G-invariant open neighbourhood N of the zero section C in E and a G-equivariant diffeomorphism Φ of N onto a G-invariant open neighbourhood of C in M restricting to the identity map on the zero section. We denote the G-equivariant pushed-forward

projection $\tau \circ \Phi^{-1}$ by $\tilde{\tau}$. We shall postpone a sketch of an existence proof of the G-invariant tubular neighbourhood at the end of this proof.

Consider now a G-invariant smooth partition of unity $\psi_U + \psi_W = 1$ subordinate to the G-invariant open covering of M by U and $W := M \setminus C$ (which exists thanks to the properness of the G-action, see [41, p. 78, Thm. 5.2.5.]). For any $\phi \in C^\infty(C)$ define $\mathrm{prol}\,\phi$ to be equal to $\psi_U(\phi \circ \tilde{\tau})$ on U and zero outside of U. Clearly, prol is G-equivariant and satisfies the asserted properties.

The existence proof of a G-invariant tubular neighbourhood can largely be copied from the case $G = \{e\}$ in Lang's book [34, p. 108–110] by observing the following additional facts:

Thanks to the properness of the G-action there is a G-invariant Riemannian metric on M (see [40, p. 316, Thm 4.3.1.]) inducing a G-equivariant vector bundle isomorphism of E onto the Riemannian normal bundle of C in $TM|_C$. Φ can then be defined as this morphism followed by the exponential map of the metric, which can easily be seen to be a well-defined G-equivariant local diffeomorphism of a G-invariant open neighbourhood of C in E onto a G-invariant open neighbourhood of C in M.

It is more difficult to make Φ injective on a possibly smaller G-invariant open neighbourhood of C in E: suppose that for each $c \in C$ there is a *G-invariant open neighbourhood* restricted to which Φ is injective (a fact which we shall show further down). Then the method of patching together local inverses (which are necessarily G-equivariant) explained in Lang's book (following an argument by Godement) can be transferred to our case since by properness of the G-action the quotient space M/G is still paracompact (although in general no longer a manifold), see [40, p. 302, Prop. 1.2.8. and p. 316, Thm 4.3.1.]: this guarantees the existence of G-invariant locally finite coverings allowing for G-invariant shrinkings and closures making all sets appearing in the standard tubular neighbourhood proof G-invariant.

Finally, suppose that there were a point $c \in C$ having no G-invariant open neighbourhood in E restricted to which Φ is injective. Then there would be a sequence (X_n) of open neighbourhoods of c having compact closure and intersection $\{c\}$, two sequences $a_n, b_n \in X_n \setminus \{c\}$ converging both to c, and a sequence $g_n \in G$ such that $g_n a_n \neq b_n$ but $g_n \Phi(a_n) = \Phi(g_n a_n) = \Phi(b_n)$ for all positive integers n (using the open G-invariant sets $\{gx | g \in G, x \in X_n\}$). Since G properly acts on M we can assume (by possibly restricting to a subsequence) that g_n converges to $g \in G$ with $gc = c$, hence $g_n b_n$ converges to c. But then there is an integer n_0 such that a_{n_0} and $g_{n_0} b_{n_0}$ both are in a (not necessarily G-invariant) tubular neighbourhood N of C in E restricted to which Φ is injective which is a contradiction. □

A prolongation constructed in the above proof will be called a *geometric* prolongation.

Let us suppose from now on that the manifold M is *symplectic* with symplectic form ω, and that $(M, \omega, \mathfrak{g}, J)$ is a Hamiltonian $\mathfrak{g}$-space. In the rest of this paper we shall very often encounter the situation that 0 is a regular value of the equivariant momentum map J and that the *constraint surface* $C := J^{-1}(\{0\})$ is nonempty in which case we shall henceforth call the quintuple $(M, \omega, \mathfrak{g}, J, C)$ a *Hamiltonian $\mathfrak{g}$-space with regular constraint surface* which in physics is often called the irreducible case. The vanishing ideal $\mathcal{I}_C$ is a Poisson sub-algebra of $C^\infty(M)$, which is equivalent to C being a coisotropic ('first class') sub-manifold. Let $\mathcal{B}_C$ denote the *normalizer* of the vanishing ideal, i.e. the space of all those functions in $C^\infty(M)$ whose Poisson bracket with every function in the vanishing ideal is again contained in the vanishing ideal. Then $\mathcal{I}_C$ is a Poisson ideal in $\mathcal{B}_C$ and the quotient $\mathcal{B}_C/\mathcal{I}_C$ becomes a Poisson algebra, see e.g. [31, p. 443], [1, p. 417–418] or Glößner's paper [25] for a proof. For Hamiltonian G-spaces we shall speak of *Hamiltonian G-spaces with regular constraint surface* if the corresponding Hamiltonian $\mathfrak{g}$-space has a regular constraint surface. In this case it is known that the group action on C has zero-dimensional isotropy groups. In order to define a smooth manifold structure on the reduced phase space $M_{\mathrm{red}} := C/G$ which will turn the canonical projection $\pi : C \to M_{\mathrm{red}}$ into a smooth open submersion, the group action does not have to be proper on all of M but has to be 'sufficiently nice', e.g. proper and free, on the constraint surface C only. The following description of the space of smooth complex-valued functions on the reduced space is well-known:

Proposition 4 *Let $(M, \omega, \mathfrak{g}, J, C)$ be a Hamiltonian G-space with regular constraint surface, such that the connected Lie group G acts in a sufficiently nice way (e.g. freely and properly) on C such*

that M_{red} exists and the canonical projection π is a smooth open submersion.

i.) The maps $\pi^* : C^\infty(M_{\mathrm{red}}) \to C^\infty(C)$ and $\iota^* : C^\infty(M) \to C^\infty(C)$ induce the following bijections on the space of all G-invariant functions $C^\infty(C)^G$

$$\pi^* : C^\infty(M_{\mathrm{red}}) \xrightarrow{\simeq} C^\infty(C)^G, \qquad \iota^* : \mathcal{B}_C/\mathcal{I}_C \xrightarrow{\simeq} C^\infty(C)^G. \tag{6}$$

ii.) For any chosen prolongation prol *one has* $\mathrm{prol}\,\pi^* C^\infty(M_{\mathrm{red}}) \subset \mathcal{B}_C$. *Moreover, the space of all G-invariant functions $C^\infty(M)^G$ on M is contained in $\mathcal{B}_C$.*

In case the group action is proper on all of M we have that $\iota^* C^\infty(M)^G = \iota^* \mathcal{B}_C$.

iii.) Suppose that the Lie group G acts properly and freely on C Then there is a left inverse $\sigma : C^\infty(C) \to C^\infty(M_{\mathrm{red}})$ of π^*, i.e. $\sigma\pi^* = \mathsf{id}_{C^\infty(M_{\mathrm{red}})}$ (this is the 'gauge fixing map').

iv.) The Poisson bracket of two functions $\phi_1, \phi_2 \in C^\infty(M_{\mathrm{red}})$ can be written as

$$\{\phi_1, \phi_2\}_{\mathrm{red}} = \sigma\iota^*\{\mathrm{prol}\,\pi^*\phi_1, \mathrm{prol}\,\pi^*\phi_2\} \quad \textit{or} \quad \pi^*\{\phi_1, \phi_2\}_{\mathrm{red}} = \iota^*\{\mathrm{prol}\,\pi^*\phi_1, \mathrm{prol}\,\pi^*\phi_2\}. \tag{7}$$

Proof: The first point is a direct consequence of the definitions and of Lemma 2. To prove that any function in $C^\infty(M)^G$ lies in $\mathcal{B}_C$ we can use an explicit description of $\mathcal{I}_C$ generated by $\langle J, \xi \rangle$, $\xi \in \mathfrak{g}$ (a result which will be proved in Lemma 5). The rest of this point is a consequence of Lemma 3. To prove the third point observe that C is a principal fibre bundle over the reduced space. Choose a locally finite open covering $(U_\alpha)_{\alpha\in I}$ of M_{red} over which the bundle is trivial, choose local sections $\sigma_\alpha : U_\alpha \to C$ of this bundle, and a smooth partition of unity $(\psi_\alpha)_{\alpha\in I}$ subordinate to the covering. Then the map $\sigma(\phi) := \sum_{\alpha\in I} \psi_\alpha(\sigma_\alpha^*\phi)$ will do the job. In case the fibres are compact one may also integrate out the fibres with respect to some density to get the desired σ. The last part is a direct computation. □

Note that for non-proper group actions it is in general no longer true that each smooth complex-valued function on the reduced phase space is induced by a globally G-invariant function as the following example shows:

Let M be the cotangent bundle of the two-torus T^2 minus its zero-section. It is diffeomorphic to the Cartesian product $T^2 \times (\mathbb{R}^2 \setminus \{0\})$. Let $G = \mathbb{R}$ whence its Lie algebra and its dual are canonically isomorphic to $\mathbb{R}$. Let $J : M \to \mathbb{R}$ be the function $J(z_1, z_2, p_1, p_2) := \frac{1}{2}(p_1^2 - p_2^2)$. It is easy to see that every value of J is regular, but that the orbit space of the corresponding constraint surface is a smooth Hausdorff manifold if and only if the value is equal to 0: in this case the two angular frequencies $\partial J/\partial p_1$ and $\partial J/\partial p_2$ are always rationally dependent so that the reduced phase space is symplectomorphic to two copies of the cotangent bundle of the unit circle minus the zero-section. In all the other cases a generic G-orbit is not closed, but has a closure diffeomorphic to T^2. From this it easily follows that every globally G-invariant smooth complex-valued function f on M is of the general form $f(z_1, z_2, p_1, p_2) = \phi(p_1, p_2)$. Hence its restriction to the constraint surface $C := J^{-1}(\{0\})$ does clearly only induce those functions on $T^*S^1 \setminus S^1$ which are invariant under $U(1)$-rotations. In this example the difference between the so-called 'strongly invariant functions' (the elements of $C^\infty(M)^G$) and the 'weakly invariant functions' (the elements of $\mathcal{B}_C$) becomes crucial.

Finally we should like to mention that the Marsden-Weinstein reduction for Hamiltonian G-spaces for a *non-zero value* μ of the momentum map can be reduced to the above case by adding a suitable coadjoint orbit, see e.g. [26, p. 194, Thm. 26.6]. Note also that the G-action on the extended system is proper if the original action was proper.

We shall give another description of the space $C^\infty(M_{\mathrm{red}})$ as the zero group of BRST cohomology in the next section. To this end let us recall briefly some results on Koszul and Chevalley-Eilenberg cohomology related to phase space reduction.

Let V be an n-dimensional real vector space, V^* its dual, and $J : M \to V^*$ a smooth map such that 0 is a regular value of J and the constraint surface $C := J^{-1}(\{0\})$ is nonempty. Then C is a regular sub-manifold of codimension n of M. Define the *ideal generated by J*, $\mathcal{I}(J)$, as the ideal of $C^\infty(M)$ spanned by all functions of the form $f\langle J, \xi\rangle$ where $\xi \in V, f \in C^\infty(M)$. Note that this definition also makes sense for any smooth map $M \to V^*$. Denote by $\bigwedge V$ the Graßmann algebra over V and consider the tensor product $\bigwedge V \otimes C^\infty(M)$. Let

$$\partial : \bigwedge^\bullet V \otimes C^\infty(M) \to \bigwedge^{\bullet-1} V \otimes C^\infty(M), \quad a \mapsto i(J)a \tag{8}$$

denote the *Koszul boundary operator* associated to C and J. Here $i(J)$ means the left insertion (the standard interior product) of J. We shall sometimes write ∂_i, $1 \leq i \leq n$, for the restriction of ∂ to $\bigwedge^i V \otimes C^\infty(M)$. The pair $(\bigwedge V \otimes C^\infty(M), \partial)$ becomes a chain complex (as $\partial^2 = 0$). For a regular constraint surface this complex is known to be acyclic, which can be seen using an augmentation (see e.g. [30, Def. 6.5, p. 339]): Let $(C^\infty(C) \oplus (\bigwedge V \otimes C^\infty(M)), \hat{\partial})$ be the *augmented Koszul complex*, where $\hat{\partial}$ is defined by $\hat{\partial}_i := \partial_i$ for $1 \leq i \leq n$ and $\hat{\partial}_0 := \iota^*$ (the *augmentation*).

Lemma 5 *With the above notations suppose that the constraint surface is a regular value of the map J. Then there is a chain homotopy for the augmented complex: More precisely there is a linear map $\hat{h}$ with components $\hat{h}_{-1} = \mathrm{prol} : C^\infty(C) \to C^\infty(M)$, $\hat{h}_i = h_i : \bigwedge^i V \otimes C^\infty(M) \to \bigwedge^{i+1} V \otimes C^\infty(M)$, for $0 \leq i \leq n$, such that $\hat{h}\hat{\partial} + \hat{\partial}\hat{h} = \mathsf{id}$. Moreover, we can choose h_0 such that*

$$h_0 \,\mathrm{prol} = 0. \tag{9}$$

In particular, the vanishing ideal $\mathcal{I}_C$ is equal to the space of Koszul-0-boundaries which in turn is equal to the ideal generated by J, $\mathcal{I}(J)$.

The additional technical equation (9) will become rather useful for the quantum deformation of all this in Section 6. Again, for (proper) group actions there is an equivariant analogue:

Lemma 6 *Under the above circumstances suppose in addition that a Lie group G acts on M leaving invariant C. Suppose furthermore that there is a representation of G in V such that J is an equivariant map (with respect to the contragredient representation of G on V^*). Then all the maps $\hat{\partial}_i$ are equivariant with respect to the natural action of G on $\bigwedge V \otimes C^\infty(M)$ and $C^\infty(C)$. Moreover, the chain homotopy $\hat{h}$ can in addition be chosen to be an equivariant map in case the group action is proper.*

PROOF: Again we shall only be treating the G-invariant case and use the notation of the proof of Lemma 3. As in [23, p. 9–10] we shall first construct the chain homotopies separately on a G-invariant open neighbourhood U of C and on an open G-invariant set W not meeting C such that $U \cup W = M$. The overlap region $U \cap W$ has to be treated with care to ensure equation (9).

1. Start with a G-invariant tubular neighbourhood U' as constructed in Lemma 3 with G-equivariant projection $\tilde{\tau}$. Using the fact that C is a regular constraint surface and techniques analogous to the ones used in Lemma 3 to establish injectivity of Φ we can arrange things in such a way that the map $U' \to C \times V^* : u \mapsto (\tilde{\tau}(u), J(u))$ becomes a G-equivariant diffeomorphism onto an open G-invariant neighbourhood Z of C in $C \times V^*$ when restricted to a suitable G-invariant open neighbourhood $U \subset U'$ of C in M which we shall often identify with its image Z in the sequel. Shrinking U if necessary in a G invariant manner allows us to assume that for each point $(c, \mu) \in Z$ the interval $\{(c, t\mu) | t \in [0, 1]\}$ is also contained in Z. Using a basis $e_1, \ldots, e_n$ of V and linear co-ordinates $\alpha_1, \ldots, \alpha_n$ on V^* we define the map $h_U : \bigwedge^\bullet V \otimes C^\infty(U) \to \bigwedge^{\bullet+1} V \otimes C^\infty(U)$ by

$$h_U(\phi)(c, \mu) := \sum_{i=1}^n e_i \wedge \int_0^1 t^k \frac{\partial \phi}{\partial \alpha_i}(c, t\mu) dt, \tag{10}$$

where $\phi \in \bigwedge^k V \otimes C^\infty(U)$. It is a routine check (similar to the proof of the Poincaré Lemma upon noting that J is equal to the projection of the second factor in $C \times V^*$) that h_U is a chain homotopy for the restriction of the Koszul boundary operator to $\bigwedge V \otimes C^\infty(U)$ and that $\tau_U^* \iota^* + \partial h_U = \mathsf{id}_{C^\infty(U)}$ where $\tau_U := \tilde{\tau}|_U$.

2. Let W be the complement of the closure of the set of all those points in U whose second co-ordinate in Z is multiplied by $1/2$. W is a G-invariant open subset of M such that $M = U \cup W$. Let $1 = \psi_U + \psi_W$ be a G-invariant smooth partition of unity subordinate to that covering. We shall show in the next subsection that there is G-equivariant smooth map $\xi : W \to V$ with support in W such that i) $\langle J, \xi \rangle = \psi_W$ and ii) $\xi|_{\mathrm{supp}(\psi_U) \cap W} = -h_U(\psi_U)$. Defining for each $\phi \in \bigwedge V \otimes C^\infty(W)$ the map $h_W(\phi) := \xi \wedge \phi$ it is not hard to check that $h(\phi) := \psi_U h_U(\phi|_U) + h_W(\phi|_W)$ is the desired chain homotopy satisfying (9) for the geometric G-equivariant prolongation map constructed in Lemma 3 using ψ_U.

3. On $U \cap W$ define $\hat{\xi} := -h_U(\psi_U)$ which is clearly G-invariant and satisfies i) and ii) above. Let W' be the G-invariant open set $W \setminus \mathrm{supp}(\psi_U)$. In order to define ξ on W' with property i) above we proceed as follows: by the properness of the G-action the isotropy subgroup G_x of each point $x \in W'$ is compact. Using a G_x-invariant scalar product on V^* it is easy to construct a smooth G-equivariant map $\eta^{[x]}$ of the G-orbit through x into V^* satisfying $\langle J, \eta^{[x]} \rangle = 1 \quad \forall x \in W'$. Again by the properness of the action each orbit is closed, and upon using a locally finite system of sufficiently small G-invariant tubular neighbourhoods around each orbit with subordinate G-invariant smooth partition of unity and G-equivariant prolongation maps (see again Lemma 3) we can glue together the prolongations of the maps $\eta^{[x]}$ to a G-equivariant smooth map $\xi' : W' \to V^*$ satisfying property i). The map ξ is obtained by glueing $\hat{\xi}$ on $U \cap W$ and ξ' on W' by means of a G-invariant smooth partition of unity subordinate to the covering of W by $U \cap W$ and W'. $\square$

Again we shall call chain homotopies constructed in the above proof *geometric* chain homotopies.

Let $\rho : \mathfrak{g} \to \mathsf{Hom}(Q, Q)$ be a representation of the finite-dimensional Lie algebra $\mathfrak{g}$ in some complex vector space Q. Recall the definition of the *Chevalley-Eilenberg differential* $\delta : \bigwedge^\bullet \mathfrak{g}^* \otimes Q \to \bigwedge^{\bullet+1} \mathfrak{g}^* \otimes Q$: let $\alpha \otimes q \in \bigwedge^k \mathfrak{g}^* \otimes Q$ and $\xi_1, \ldots, \xi_{k+1} \in \mathfrak{g}$, then

$$\begin{aligned} \delta(\alpha \otimes q)(\xi_1, \ldots, \xi_{k+1}) \quad := \quad & \sum_{1 \le i < j \le k+1} (-1)^{i+j+1} \alpha([\xi_i, \xi_j], \xi_1, \ldots, \overset{i}{\wedge}, \ldots, \overset{j}{\wedge}, \ldots, \xi_{k+1}) \otimes q \\ & + \sum_{i=1}^{k+1} (-1)^{i+1} \alpha(\xi_1, \ldots, \overset{i}{\wedge}, \ldots, \xi_{k+1}) \otimes \rho(\xi_i)(q). \end{aligned} \tag{11}$$

It is well-known that $\delta^2 = 0$. We shall denote by $H^\bullet_{\mathrm{CE}}(\mathfrak{g}, Q)$ the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module Q for the quotient of the kernel of δ by the image of δ. This space clearly inherits the $\mathbb{Z}$-grading of the Graßmann algebra over $\mathfrak{g}^*$.

For computations we shall frequently use a basis $e_1, \ldots, e_n$ ($n := \dim \mathfrak{g}$) of $\mathfrak{g}$ and its dual base $e^1, \ldots, e^n$ of $\mathfrak{g}^*$. Denoting by $f^c{}_{ab} := \langle e^c, [e_a, e_b] \rangle$ the structure constants of $\mathfrak{g}$ we get the following short formula for δ:

$$\delta(\alpha \otimes q) = -\frac{1}{2} \sum_{a,b,c} f^c{}_{ab}\, e^a \wedge e^b \wedge i(e_c)\alpha \otimes q + \sum_a e^a \wedge \alpha \otimes \rho(e_a)(q) \tag{12}$$

Recall that the zeroth Chevalley-Eilenberg cohomology group $H^0_{\mathrm{CE}}(\mathfrak{g}, Q)$ is always equal to the space of $\mathfrak{g}$-*invariants* $Q^{\mathfrak{g}} := \{q \in Q \mid \rho(\xi) q = 0\}$.

The significance of the Chevalley Eilenberg differential becomes clear by the following obvious characterisation of the space of smooth complex-valued functions on the reduced phase space:

Proposition 7 *Let (M, ω, G, J, C) be a Hamiltonian G-space with regular constraint surface such that in addition the connected Lie group G acts properly and freely on the constraint surface C. Then the space $C^\infty(M_{\mathrm{red}})$ is canonically isomorphic to the zeroth Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in $\mathfrak{g}$-module $C^\infty(C)$ (with representation ϱ_C, see (2)) via the pull-back π^*. For a general Hamiltonian Lie algebra action the space $H^0_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C))$ will be called the space of classical invariants on the constraint surface C.*

4 The classical BRST cohomology with augmentation

In this section we recall the construction of the classical BRST cohomology following [33]. Throughout the section $(M, \omega, \mathfrak{g}, J)$ is a Hamiltonian $\mathfrak{g}$-space. The classical BRST complex is a double complex whose total complex is also a differential graded Poisson algebra, i.e. the total differential is a super-derivation of a super Poisson structure. It is constructed as follows:

i.) The space of chains is the $\mathbb{Z} \times \mathbb{Z}$-graded vector space $\mathcal{A}^{\bullet,\bullet} := \bigwedge^\bullet \mathfrak{g}^* \otimes \bigwedge^\bullet \mathfrak{g} \otimes C^\infty(M)$, where the gradings are called by convention *ghost* and *antighost degree* (following [29, p. 191] where these gradings are called 'pure ghost number' and 'antighost number'). Moreover, $\mathcal{A}$ carries a natural $\mathbb{Z}_2$-graded vector space structure $\mathcal{A} = \bigwedge^{\mathrm{even}}(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes C^\infty(M) \oplus \bigwedge^{\mathrm{odd}}(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes C^\infty(M)$ and a $\mathbb{Z}$-grading $\mathcal{A}^{(n)} = \bigoplus_{n=k-l} \mathcal{A}^{k,l}$, where $n \in \mathbb{Z}$ is by convention called *ghost number*, see [29, p. 191] (also called 'total degree' in [33, p. 57]). Using the $\wedge$-product of forms (which of course is graded in the standard way, i.e. $(\alpha \otimes \xi) \wedge (\beta \otimes \eta) = (-1)^{kl} (\alpha \wedge \beta) \otimes (\xi \wedge \eta)$, where $\alpha, \beta \in \bigwedge \mathfrak{g}^*$, $\xi, \eta \in \bigwedge \mathfrak{g}$, and the degrees of β and ξ are k, l, respectively) and the pointwise product of functions, $\mathcal{A}$ becomes an associative, super-commutative algebra, graded with respect to all the above mentioned degrees. We shall sometimes use the physicist's terminology to call elements of $\bigwedge \mathfrak{g}^*$ ghosts and elements of $\bigwedge \mathfrak{g}$ antighosts.

ii.) The vertical differential is taken to be the standard Chevalley-Eilenberg differential $\delta : \mathcal{A}^{\bullet,\bullet} \to \mathcal{A}^{\bullet+1,\bullet}$ of the Lie algebra cohomology of $\mathfrak{g}$ with respect to the $\mathfrak{g}$-module $\bigwedge \mathfrak{g} \otimes C^\infty(M)$ where the representation is given by $\mathfrak{g} \ni \xi \mapsto \mathrm{ad}(\xi) \otimes \mathsf{id} + \mathsf{id} \otimes \{\langle J, \xi \rangle, \cdot\}$, see (11). Its cohomology is denoted by $H^\bullet_{\mathrm{CE}}(\mathcal{A})$.

iii.) The horizontal differential $\partial : \mathcal{A}^{\bullet,\bullet} \to \mathcal{A}^{\bullet,\bullet-1}$ is defined to be the standard extension of the previously defined Koszul differential (8) to the complex $\mathcal{A}$, i.e. $\partial(\alpha \otimes x \otimes F) = (-1)^k \alpha \otimes i(J)(x \otimes F)$ for $\alpha \in \bigwedge^k \mathfrak{g}^*$, $x \in \bigwedge \mathfrak{g}$, and $F \in C^\infty(M)$. Its homology will be denoted by $H^{\mathrm{Kos}}_\bullet(\mathcal{A})$.

iv.) It is easy to see that ∂ anti-commutes with δ and hence we form out of this double complex its total complex as follows. The total differential $\mathcal{D} : \mathcal{A}^{(\bullet)} \to \mathcal{A}^{(\bullet+1)}$ is taken to be

$$\mathcal{D} := \delta + 2\partial \tag{13}$$

and is called the *(classical) BRST operator* (this inessentially differs from [33] where the BRST operator was defined to be $\delta + 2(-)^k \partial$, where k is the antighost degree).

v.) The algebra $\mathcal{A}$ has a natural super Poisson structure induced by the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ and the Poisson bracket on M. In order to describe this bracket we firstly recall the definition of the left and right insertion maps on $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. Let $\alpha \in \bigwedge^k(\mathfrak{g}^*)$, $\xi \in \bigwedge^l(\mathfrak{g})$, $\beta \in \mathfrak{g}^*$, and $X \in \mathfrak{g}$. Then $i(\beta)(\alpha \otimes \xi) := (-1)^k \alpha \otimes (i(\beta)\xi)$ and $i(X)(\alpha \otimes \xi) := (i(X)\alpha) \otimes \xi$. Moreover the *right insertion* $j(v)$ for $v \in \mathfrak{g}^* \oplus \mathfrak{g}$ is defined by $j(v)a = -(-1)^m i(v)a$ where $a \in \bigwedge^m(\mathfrak{g}^* \oplus \mathfrak{g})$. Then we define the following endomorphisms P and P^* of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ (here the tensor product is not graded)

$$P := \sum_{a=1}^n j(e_a) \otimes i(e^a) \qquad \text{and} \qquad P^* := \sum_{a=1}^n j(e^a) \otimes i(e_a), \tag{14}$$

which clearly do not depend on the choice of the basis. Then the super Poisson bracket on $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ is defined by

$$\{a, b\} = 2\mu \circ (P + P^*)(a \otimes b) = 2 \sum_{c=1}^n \left(j(e_c)a \wedge i(e^c)b + j(e^c)a \wedge i(e_c)b \right), \tag{15}$$

where μ denotes the $\wedge$-product of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. The factor 2 is by convention to get the correct Clifford algebra later. Now we tensor this with the Poisson bracket on M

$$\{a \otimes F, b \otimes G\} := a \wedge b \otimes \{F, G\} + \{a, b\} \otimes FG, \tag{16}$$

for $a, b \in \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ and $F, G \in C^\infty(M)$, to get a super Poisson bracket on $\mathcal{A}^{(\bullet)}$. Note that this super Poisson bracket is still $\mathbb{Z}_2$-graded, but no longer graded with respect to ghost and antighost degree separately. It is, however, graded with respect to ghost number: this can also be seen by the Hamiltonian form of the ghost number derivation which we shall give further down.

vi.) The classical BRST operator turns out to be a Hamiltonian super-derivation of the above Poisson structure: regarding the Lie bracket $[\cdot, \cdot]$ of $\mathfrak{g}$ in a canonical manner as an element in $\bigwedge^2 \mathfrak{g}^* \otimes \mathfrak{g} \subset \mathcal{A}$, we define

$$\Omega := -\frac{1}{2}[\cdot, \cdot].$$

More precisely, this means $\Omega = -\frac{1}{4}\sum_{a,b,c=1}^n {f^c}_{ab}\, e^a \wedge e^b \wedge e_c$ in terms of a chosen basis. On the other hand we have the momentum map $J \in \mathfrak{g}^* \otimes C^\infty(M) \subset \mathcal{A}$. So

$$\Theta := \Omega + J \tag{17}$$

is an odd element in $\mathcal{A}^{(1)}$ which we call the (classical) *BRST charge*. One can easily verify that

$$\mathcal{D} = \{\Theta, \cdot\}. \tag{18}$$

vii.) Let γ be the identity endomorphism of $\mathfrak{g}$ regarded as an element of $\mathcal{A}^{1,1}$. This takes the form

$$\gamma = \frac{1}{2}\sum_{a=1}^n e^a \wedge e_a \tag{19}$$

in terms of the aforementioned basis and dual basis. Note that the ghost number grading is induced by the *ghost number derivation*

$$\mathsf{Gh} := \{\gamma, \cdot\}. \tag{20}$$

Definition 8 *The differential graded Poisson algebra $(\mathcal{A}^{(\bullet)}, \mathcal{D}, \{\cdot, \cdot\})$ is called the classical BRST algebra. Its cohomology group $\ker \mathcal{D} \big/ \operatorname{im} \mathcal{D}$ will be called the classical BRST cohomology and will be denoted by $H^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A})$.*

Lemma 9 *The classical BRST cohomology is equipped with a natural $\mathbb{Z}$-graded super Poisson structure induced by the super Poisson structure of the classical BRST algebra: let $a, b \in \mathcal{A}$ such that $\mathcal{D}a = 0 = \mathcal{D}b$; then for the corresponding cohomology classes $[a], [b] \in H^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A})$ we have $[a] \wedge [b] := [a \wedge b]$ and $\{[a], [b]\} := [\{a, b\}]$.*

Proof: Since the classical BRST operator is a super-derivation it immediately follows that its kernel is a sub-algebra of the super Poisson algebra $\mathcal{A}$. Moreover $\mathcal{D}^2 = 0$ entails that the image of $\mathcal{D}$ is a super Poisson ideal in the kernel. The grading is inherited by the induced ghost number derivation. This proves the lemma. $\square$

We shall now suppose that $(M, \omega, \mathfrak{g}, J, C)$ is a Hamiltonian $\mathfrak{g}$-space with regular constraint surface C. In order to get more information about the classical BRST cohomology and its relation to the constraint surface C we shall extend the augmentation of the previous section to the above double complex:

For any $\xi \in \mathfrak{g}$ let ξ_M and ξ_C be the (infinitesimal) generator of the action of $\mathfrak{g}$ or G on M and C, respectively. The representation ϱ_C uniquely defines the Chevalley-Eilenberg complex $(\bigwedge \mathfrak{g}^* \otimes C^\infty(C), \delta^c)$ (as, of course, the representation ϱ_M defines the Chevalley-Eilenberg operator δ on $\bigwedge \mathfrak{g}^* \otimes C^\infty(M)$). Now extend the restriction map $\iota^* : C^\infty(M) \to C^\infty(C)$ to the Chevalley-Eilenberg complex $\bigwedge \mathfrak{g}^* \otimes C^\infty(M)$ by $\iota^*(\alpha \otimes f) := (-1)^k \alpha \otimes \iota^* f$ where $\alpha \in \bigwedge^k \mathfrak{g}^*$ and $f \in C^\infty(M)$. Thanks to the identity

$$\varrho_C(\xi)\iota^* = \iota^* \varrho_M(\xi) \tag{21}$$

it is clear that

$$\delta^c \iota^* = -\iota^* \delta. \tag{22}$$

For every prolongation map prol, we have the following equation, which will become important for deformation:

$$\varrho_C(\xi) = \iota^* \varrho_M(\xi)\, \mathrm{prol} \tag{23}$$

Let $\hat{\mathcal{A}}$ denote the *augmented classical BRST complex* $(\bigwedge \mathfrak{g}^* \otimes C^\infty(C)) \oplus \mathcal{A}$, and denote by $\hat{\mathcal{D}}$ the *augmented classical BRST operator*

$$\hat{\mathcal{D}} := \delta^c + 2\iota^* + \mathcal{D}, \tag{24}$$

where all the maps are defined to be zero on the domains on which they were previously not defined. Clearly

$$\hat{\mathcal{D}}^2 = 0. \tag{25}$$

Moreover, we need to extend the chain homotopies h and prol of the classical (augmented) Koszul complex (compare Section 3, Lemma 5) to $\mathcal{A}$ and $\hat{\mathcal{A}}$ which is done in the usual way by $h(\alpha \wedge f) := (-1)^k \alpha \wedge hf$ and $\mathrm{prol}(\alpha \wedge \phi) := (-1)^k \alpha \wedge \mathrm{prol}\, \phi$, where $\alpha \in \bigwedge^k \mathfrak{g}^*$, $f \in \bigwedge \mathfrak{g} \otimes C^\infty(M)$, $\phi \in C^\infty(C)$. We keep the notation $\hat{h}$ for $\mathrm{prol} + h$ on $\hat{\mathcal{A}}$. Moreover, let $\hat{\delta}$ denote the augmented Chevalley-Eilenberg operator $\delta^c + \delta$.

The cohomology of the classical BRST complex can be computed in terms of the Chevalley-Eilenberg cohomology on the constraint surface:

Proposition 10 *Let $(M, \omega, \mathfrak{g}, J, C)$ be a Hamiltonian $\mathfrak{g}$-space with regular constraint surface. With the above notations and definitions we have:*

i.) The following map

$$\hat{h}' := \frac{1}{2}\hat{h}\left(\mathsf{id} + \frac{1}{2}(\hat{\delta}\hat{h} + \hat{h}\hat{\delta})\right)^{-1} \tag{26}$$

is a chain homotopy for the augmented complex, i.e. $\hat{\mathcal{D}}\hat{h}' + \hat{h}'\hat{\mathcal{D}} = \mathsf{id}$*. The linear map*

$$\Psi : H^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}) \to H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^{\infty}(C)) : \quad [a] \mapsto [\iota^* a] \tag{27}$$

is an isomorphism with the following inverse (where $[c] \in H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^{\infty}(C))$*):*

$$\Psi^{-1} : [c] \mapsto \left[2\hat{h}'c\right] = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \left[(h\delta)^k \operatorname{prol} c\right]. \tag{28}$$

ii.) The isomorphism Ψ *turns the Chevalley-Eilenberg cohomology on the constraint surface,* $H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^{\infty}(C))$*, into a* $\mathbb{Z}$*-graded super Poisson algebra. The exterior multiplication and the super Poisson bracket take the following form for* $c_1, c_2 \in \bigwedge\mathfrak{g}^* \otimes C^{\infty}(C)$*:*

$$[c_1] \wedge [c_2] := [c_1 \wedge c_2], \tag{29}$$

$$\{[c_1], [c_2]\} := [\iota^*\{\operatorname{prol} c_1, \operatorname{prol} c_2\}] - \frac{1}{2}\,[\iota^*\{\operatorname{prol} c_1, h_0\delta \operatorname{prol} c_2\}] - \frac{1}{2}\,[\iota^*\{h_0\delta \operatorname{prol} c_1, \operatorname{prol} c_2\}]\,. \tag{30}$$

In particular for the classical invariants, i.e. the elements of $H^{0}_{\mathrm{CE}}(\mathfrak{g}, C^{\infty}(C))$*, the Poisson bracket reduces to*

$$\{[c_1], [c_2]\} = [\iota^*\{\operatorname{prol} c_1, \operatorname{prol} c_2\}]\,. \tag{31}$$

PROOF: For the first assertion, note that the chain homotopy equation $\hat{h}\hat{\partial} + \hat{\partial}\hat{h} = \mathsf{id}$ of the 'pure' augmented Koszul complex (see Lemma 5) still holds on the augmented BRST complex. Hence $\hat{\mathcal{D}}\hat{h} + \hat{h}\hat{\mathcal{D}} = 2\mathsf{id} + \hat{\delta}\hat{h} + \hat{h}\hat{\delta}$. Since $\hat{\delta}\hat{h} + \hat{h}\hat{\delta}$ is obviously nilpotent of order at most $n+1$ the right hand side is invertible. Thanks to $\hat{\mathcal{D}}^2 = 0$ the map $\hat{\delta}\hat{h} + \hat{h}\hat{\delta}$ commutes with $\hat{\mathcal{D}}$. Thus $\hat{h}'$ is well-defined and a chain homotopy. The fact that the map (27) is well-defined and bijective is shown using the chain homotopy property and the equations $2\iota^*\hat{h}'\iota^* = \iota^*$ and $\mathcal{D}\hat{h}'\iota^* + \hat{h}'\delta^c\iota^* = 0$ which straightforwardly follow from the definitions by pure diagram chase. The simplification of Ψ^{-1} is a simple consequence of c being closed with respect to δ^c. The second assertion is clear by Lemma 9. The form of the exterior multiplication is quickly computed using the fact that $\mathcal{A}$ is $\mathbb{Z} \times \mathbb{Z}$-graded as a super-commutative algebra. Note that the exterior multiplication of the classes is well-defined since the representation ϱ_C is a derivation on the $\mathfrak{g}$-module $C^{\infty}(C)$. □

We shall finally discuss the case when the Hamiltonian $\mathfrak{g}$-space comes from a Hamiltonian G-space where the connected Lie group G acts properly on M and freely on C: We can now choose the chain homotopies equivariant under the G-action (see Lemma 6). Consequently

$$\hat{\delta}\hat{h} + \hat{h}\hat{\delta} = 0, \tag{32}$$

and the formula for the super Poisson bracket (30) simplifies to (31) for all elements of the Chevalley-Eilenberg cohomology. Note that the BRST cohomology is computing the so-called cohomology along the leaves or G-orbits, see e.g. [16, Thm. 3.8, p. 53] for a more precise statement. Furthermore a formula for the Poisson bracket of two functions $\phi_1, \phi_2 \in C^{\infty}(M_{\mathrm{red}})$ is easily written down upon using a suitable left inverse σ of the pull-back with the projection $\pi : C \to M_{\mathrm{red}}$ (cf. Proposition 4, *iii.)*) and yields the same result as in (7).

5 The quantum BRST operator in Deformation Quantization

In this section the quantum BRST algebra will be defined and we shall describe some operators associated with it. Throughout this section $(M, *, \mathfrak{g}, \boldsymbol{J})$ will be a fixed Hamiltonian quantum $\mathfrak{g}$-space. Although our aim is to describe the case of a regular constraint surface (the irreducible case) our definitions and results in this section do not need this restriction, neither do we need the fact that M is symplectic.

The underlying vector space for the quantum BRST algebra is the $\mathbb{C}[[\lambda]]$-module $\mathcal{A}[[\lambda]]$ of formal power series with values in $\mathcal{A}$ endowed with its ghost and antighost gradings as defined in the previous section. Moreover, $\mathcal{A}[[\lambda]]$ inherits the ghost number grading from $\mathcal{A}$ as well as the $\mathbb{Z}_2$-grading in even and odd elements. Using the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$ as inner product we can define a one parameter family of equivalent products $\circ_\kappa$ for the Graßmann part of $\mathcal{A}[[\lambda]]$ indexed by a parameter $\kappa \in [0,1]$:

$$\alpha \circ_\kappa \beta = \mu \circ \mathrm{e}^{2\mathrm{i}\lambda(\kappa P + (1-\kappa)P^*)} \alpha \otimes \beta, \tag{33}$$

where the operators P and P^* are given as in (14). This multiplication is known to be associative since left and right insertions are anti-commuting super-derivations (see e.g. [5, Prop. 2.1] for a proof). Moreover, note that all the products $\circ_\kappa$ satisfy the Clifford relation on the one-forms for a multiple of the quadratic form on $\mathfrak{g}^* \oplus \mathfrak{g}$ defined by the natural pairing. All the multiplications $\circ_\kappa$ are formal associative deformations of the $\wedge$-product with first order super-commutator being $\mathrm{i}\lambda$ times the super Poisson bracket (15). Taking now one fixed κ-ordered product $\circ_\kappa$ for the Graßmann part and the quantum covariant star product $*$ for the functions we obtain an associative product for $\mathcal{A}[[\lambda]]$ by tensoring these algebra structures which we shall denoted by $\star_\kappa$: more precisely, for $\alpha \otimes F, \beta \otimes G \in \mathcal{A}[[\lambda]]$ with $\alpha, \beta \in \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})[[\lambda]]$ and $F, G \in C^\infty(M)[[\lambda]]$ we set

$$(\alpha \otimes F) \star_\kappa (\beta \otimes G) = (\alpha \circ_\kappa \beta) \otimes (F * G). \tag{34}$$

Then all $\star_\kappa$ are equivalent by means of the equivalence transformation $S_\kappa = \exp(2\mathrm{i}\kappa\lambda\Delta)$ where Δ is defined by

$$\Delta := \sum_{a=1}^{n} i(e_a) i(e^a) = -\sum_{a=1}^{n} j(e_a) j(e^a). \tag{35}$$

Clearly Δ does not depend on the choice of the basis and the equivalence

$$S_\kappa(a \circ_\kappa b) = S_\kappa a \circ_{\mathrm{S}} S_\kappa b \tag{36}$$

is a straightforward computation where $a, b \in \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. It can easily be seen that $\star_\kappa$ is still $\mathbb{Z}_2$-graded but $\star_\kappa$ is no longer graded with respect to the ghost and antighost degree separately. It will, however, be graded with respect to ghost number which can be seen by means of the classical ghost number derivation which turns out to be a derivation of $\star_\kappa$. Taking super-commutators with respect to $\star_\kappa$ will be denoted by ad_κ. In the particular cases of the Weyl (i.e. $\kappa = \frac{1}{2}$) and standard ordered (i.e. $\kappa = 0$) multiplications we shall use the notation $\star_{\mathrm{W}}$ and $\star_{\mathrm{S}}$ for the star products, respectively, and ad_{W} and ad_{S} for the super-commutators, respectively.

Now we should like to turn $\mathcal{A}[[\lambda]]$ into a *graded differential algebra* by defining a cochain complex whose differential is an odd $\mathbb{C}[[\lambda]]$-linear left super-derivation raising the ghost number by one. This is achieved in the following way: We define the *quantum Weyl ordered BRST charge* to be

$$\Theta_{\mathrm{W}} := \Omega + \boldsymbol{J} \tag{37}$$

which is odd and of ghost number 1. Associated to Θ_{W} is the *Weyl ordered BRST operator*:

$$\boldsymbol{\mathcal{D}}_{\mathrm{W}} := \frac{1}{\mathrm{i}\lambda}\mathrm{ad}_{\mathrm{W}}(\Theta_{\mathrm{W}}). \tag{38}$$

Note that $\boldsymbol{\mathcal{D}}_{\mathrm{W}}$ is well defined because $\star_{\mathrm{W}}$ is super-commutative in 0-th order of λ. Moreover $\boldsymbol{\mathcal{D}}_{\mathrm{W}}$ is odd. Under the equivalence transformations S_κ, the BRST charge and the BRST operator transform according to:

$$\boldsymbol{\mathcal{D}}_\kappa := S^{-1}_{\kappa-\frac{1}{2}} \circ \boldsymbol{\mathcal{D}}_{\mathrm{W}} \circ S_{\kappa-\frac{1}{2}}, \tag{39}$$

$$\Theta_\kappa := S^{-1}_{\kappa-\frac{1}{2}}(\Theta_{\mathrm{W}}), \tag{40}$$

such that $\boldsymbol{\mathcal{D}}_\kappa = (\mathrm{i}\lambda)^{-1}\mathrm{ad}_\kappa(\Theta_\kappa)$. The standard ordered quantum BRST operator (where $\Theta_{\mathrm{S}} = \Theta_0$)

$$\boldsymbol{\mathcal{D}}_{\mathrm{S}} := \frac{1}{\mathrm{i}\lambda}\mathrm{ad}_{\mathrm{S}}(\Theta_{\mathrm{S}}) \tag{41}$$

will turn out to be of major importance in the sequel.

Definition 11 *The triple $(\mathcal{A}^{(\bullet)}[[\lambda]], \star_\kappa, \boldsymbol{\mathcal{D}}_\kappa)$ is defined to be the κ-ordered quantum BRST algebra.*

Lemma 12 *For all $\kappa \in [0,1]$ we have:*

i.) $\Theta_\kappa = \Omega + \boldsymbol{J} + \mathrm{i}\lambda(1-2\kappa)\chi$ *where $\chi \in \mathfrak{g}^* \subset \mathcal{A}^{1,0}[[\lambda]]$ defined by $\chi(\xi) = \frac{1}{2}\mathsf{tr}(\mathrm{ad}(\xi))$ for $\xi \in \mathfrak{g}$ is the trace form of $\mathfrak{g}$.*

ii.) $\Theta_\kappa \star_\kappa \Theta_\kappa = 0$.

iii.) The classical ghost number derivation $\mathsf{Gh} = \{\gamma, \cdot\}$, see (19, 20), is equal to $\frac{1}{\mathrm{i}\lambda}\mathrm{ad}_\kappa(\gamma)$ for all $\kappa \in [0,1]$ and induces the ghost number grading: $\mathsf{Gh}(\Phi) = n\Phi \iff \Phi \in \mathcal{A}^{(n)}[[\lambda]]$ for all $\Phi \in \mathcal{A}[[\lambda]]$.

iv.) $[\mathsf{Gh}, \boldsymbol{\mathcal{D}}_\kappa] = \boldsymbol{\mathcal{D}}_\kappa$*, hence $\boldsymbol{\mathcal{D}}_\kappa$ raises the ghost number by one.*

PROOF: The first part is a simple computation using the easily verified fact that $\Delta\Omega = \chi$. For the second part, thanks to (40) it suffices to show $\Theta_{\mathrm{W}} \star_{\mathrm{W}} \Theta_{\mathrm{W}} = \Omega \circ_{\mathrm{W}} \Omega + \Omega \star_{\mathrm{W}} \boldsymbol{J} + \boldsymbol{J} \star_{\mathrm{W}} \Omega + \boldsymbol{J} \star_{\mathrm{W}} \boldsymbol{J} = 0$. We compute all these terms: Firstly notice that $P^2\Omega \otimes \Omega = 0 = (P^*)^2\Omega \otimes \Omega$. Since $PP^* = P^*P$ we only have to compute the orders λ^0, λ^1, and λ^2 of $\Omega \circ_{\mathrm{W}} \Omega$. The zeroth order is trivially zero since Ω is odd, the first order vanishes by use of the Jacobi identity for the Lie bracket of $\mathfrak{g}$. Finally the second order is shown to vanish by direct computation of $\mu \circ (P^*P\Omega \otimes \Omega) = 0$. Thus $\Omega \circ_{\mathrm{W}} \Omega = 0$. Writing $\boldsymbol{J} = \sum_c e^c \otimes \boldsymbol{J}_c$ with $\boldsymbol{J}_c \in C^\infty(M)[[\lambda]]$ using a basis of $\mathfrak{g}$ we compute the anti-commutator

$$\Omega \star_{\mathrm{W}} \boldsymbol{J} + \boldsymbol{J} \star_{\mathrm{W}} \Omega = \sum_c (\Omega \circ_{\mathrm{W}} e^c + e^c \circ_{\mathrm{W}} \Omega) \otimes \boldsymbol{J}_c = \sum_c \mathrm{i}\lambda(j(e^c)\Omega + i(e^c)\Omega) \otimes \boldsymbol{J}_c = -\frac{\mathrm{i}\lambda}{2}\sum_{a,b,c} f^c{}_{ab}\, e^a \wedge e^b \otimes \boldsymbol{J}_c.$$

Finally we have due to the quantum covariance of the star product $*$

$$\boldsymbol{J} \star_{\mathrm{W}} \boldsymbol{J} = \sum_{a,b} e^a \wedge e^b \otimes \boldsymbol{J}_a * \boldsymbol{J}_b = \frac{1}{2}\sum_{a,b} e^a \wedge e^b \otimes (\boldsymbol{J}_a * \boldsymbol{J}_b - \boldsymbol{J}_b * \boldsymbol{J}_a) = \frac{\mathrm{i}\lambda}{2}\sum_{a,b,c} f^c{}_{ab}\, e^a \wedge e^b \otimes \boldsymbol{J}_c.$$

Thus we have proved $\Theta_\kappa \star_\kappa \Theta_\kappa = 0$. The third part is again easily verified. Finally the fourth part follows from the third part: $[\mathsf{Gh}, \boldsymbol{\mathcal{D}}_\kappa] = [\mathsf{Gh}, \frac{1}{\mathrm{i}\lambda}\mathrm{ad}_\kappa(\Theta_\kappa)] = \frac{1}{\mathrm{i}\lambda}\mathrm{ad}_{\star_\kappa}(\mathsf{Gh}\Theta_\kappa) = \boldsymbol{\mathcal{D}}_\kappa$ since clearly $\mathsf{Gh}\Theta_\kappa = \Theta_\kappa$. $\square$

Proposition 13 *The κ-ordered BRST algebra is a differential graded algebra over $\mathbb{C}[[\lambda]]$ for all $\kappa \in [0,1]$. All of these are isomorphic as differential graded algebras via S_κ and are formal deformations of the classical BRST algebra (their 'classical limit').*

PROOF: It remains to compute the classical limit which is straightforward. $\square$

In order to prepare Theorem 19, in which the explicit form of the quantum BRST operator will be given, we make the following definitions:

Definition 14 *We define the following $\mathbb{C}[[\lambda]]$-linear endomorphisms of $\mathcal{A}^{\bullet,\bullet}[[\lambda]]$: For all $1 \leq k, l \leq n$, $\alpha \in \bigwedge^k \mathfrak{g}^*$, $\xi = \xi_1 \wedge \cdots \wedge \xi_l \in \bigwedge^l \mathfrak{g}$, and $F \in C^\infty(M)[[\lambda]]$ we define*

i.) $\boldsymbol{q} : \mathcal{A}^{\bullet,\bullet}[[\lambda]] \to \mathcal{A}^{\bullet,\bullet-1}[[\lambda]]$ *by*

$$\boldsymbol{q}(\alpha \wedge \xi \otimes F) := (-1)^k \alpha \wedge \sum_{i<j} (-1)^{i+j-1} [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \wedge \xi_l \otimes F. \tag{42}$$

ii.) $\boldsymbol{M}_S, \boldsymbol{M}_A : \mathcal{A}^{\bullet,\bullet}[[\lambda]] \to \mathcal{A}^{\bullet,\bullet-1}[[\lambda]]$ *by*

$$\begin{array}{rcl} \boldsymbol{M}_S(\alpha \wedge \xi \otimes F) & := & (-1)^k \sum_a \alpha \wedge i(e^a)\xi \otimes F * \boldsymbol{J}_a \\ \boldsymbol{M}_A(\alpha \wedge \xi \otimes F) & := & (-1)^k \sum_a \alpha \wedge i(e^a)\xi \otimes \boldsymbol{J}_a * F, \end{array} \tag{43}$$

where $\boldsymbol{J} = \sum_a e^a \otimes \boldsymbol{J}_a$ with $\boldsymbol{J}_a \in C^\infty(M)[[\lambda]]$.

iii.) $\boldsymbol{c} : \mathcal{A}^{\bullet,\bullet}[[\lambda]] \to \mathcal{A}^{\bullet-1,\bullet-2}[[\lambda]]$ *by*

$$\boldsymbol{c}(\alpha \wedge \xi \otimes F) := \sum_{i<j} (-1)^{i+j-1} i([\xi_i, \xi_j])\alpha \wedge \xi_1 \wedge \cdots \overset{i}{\wedge} \cdots \overset{j}{\wedge} \cdots \wedge \xi_l \otimes F. \tag{44}$$

iv.) $\boldsymbol{u} : \mathcal{A}^{\bullet,\bullet}[[\lambda]] \to \mathcal{A}^{\bullet,\bullet-1}[[\lambda]]$ *by*

$$\boldsymbol{u} := \frac{1}{\mathrm{i}\lambda} \mathrm{ad}_\kappa(\chi) = \{\chi, \cdot\}, \tag{45}$$

where $\chi \in \mathfrak{g}^*$ is the trace form as in Lemma 12.

Clearly it is sufficient to specify these operators only on the above factorising elements. Note that $\boldsymbol{u}$ does not depend on κ but consists only of the lowest order term. For later use we give for these operators the following expressions in terms of a basis of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Lemma 15 *In terms of a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ and the dual basis $e^1, \ldots, e^n$ of $\mathfrak{g}^*$ we have the following expressions:*

$$\boldsymbol{q} = -\frac{1}{2} \sum_{a,b,c} f^c{}_{ab}\, e_c \wedge i(e^a) i(e^b) \tag{46}$$

$$\boldsymbol{c} = -\frac{1}{2} \sum_{a,b,c} f^c{}_{ab}\, i(e_c) i(e^a) i(e^b) \tag{47}$$

$$\boldsymbol{u} = \sum_{a,b} f^b{}_{ab}\, i(e^a) \tag{48}$$

Proof: This is a straightforward computation. $\square$

Using the $\mathfrak{g}$-representation $\boldsymbol{\varrho}_M$ as in (4) on $C^\infty(M)[[\lambda]]$ the map

$$\mathfrak{g} \ni \xi \mapsto \mathrm{ad}(\xi) \otimes \mathsf{id} + \mathsf{id} \otimes \boldsymbol{\varrho}_M(\xi) \tag{49}$$

turns $\bigwedge\mathfrak{g} \otimes C^\infty(M)[[\lambda]]$ into a $\mathfrak{g}$-module. This motivates the following definition:

Definition 16 *i.) The Chevalley-Eilenberg differential $\mathcal{A}^{\bullet,\bullet}[[\lambda]] \to \mathcal{A}^{\bullet+1,\bullet}[[\lambda]]$ with respect to the representation (49) is denoted by $\boldsymbol{\delta}$ and called the quantised Chevalley-Eilenberg differential.*

ii.) The operator $\boldsymbol{\partial} : \mathcal{A}^{\bullet,\bullet}[[\lambda]] \to \mathcal{A}^{\bullet,\bullet-1}[[\lambda]]$ defined by

$$\boldsymbol{\partial} := \boldsymbol{M}_S + \mathrm{i}\lambda(\frac{1}{2}\boldsymbol{u} - \boldsymbol{q}) \tag{50}$$

is called the quantised Koszul differential. We shall frequently write $\boldsymbol{\partial}_i$ for its restriction to $\mathcal{A}^{\bullet,i}[[\lambda]]$, $(0 \leq i \leq n)$.

Note that the classical limits of the operators $\boldsymbol{\delta}$ resp. $\boldsymbol{\partial}$ are indeed δ resp. ∂. In the following theorem we shall relate the standard ordered BRST operator $\boldsymbol{\mathcal{D}}_{\mathrm{S}}$ to the more concrete operators $\boldsymbol{\delta}$ and $\boldsymbol{\partial}$.

Theorem 17 *The standard ordered BRST operator satisfies*

$$\boldsymbol{\mathcal{D}}_{\mathrm{S}} = \boldsymbol{\delta} + 2\boldsymbol{\partial} \tag{51}$$

and thus defines a double complex, i.e.

$$\boldsymbol{\delta}^2 = 0, \quad \boldsymbol{\partial}^2 = 0, \quad \boldsymbol{\delta}\boldsymbol{\partial} + \boldsymbol{\partial}\boldsymbol{\delta} = 0. \tag{52}$$

Proof: Clearly (52) follows from (51) and $\boldsymbol{\mathcal{D}}_{\mathrm{S}}^2 = 0$ since $\boldsymbol{\delta}$ resp. $\boldsymbol{\partial}$ are homogeneous of bidegree $(1,0)$ resp. $(0,-1)$. Thus we have to show (51) which is a straightforward computation using $\Theta_{\mathrm{S}} = \Omega + \boldsymbol{J} + \mathrm{i}\lambda\chi$. In terms of the basis we obtain by a simple computation

$$\mathrm{ad}_{\mathrm{S}}(\Omega) = \mathrm{i}\lambda\left(\sum_{a,b,c} f^c{}_{ab}\, e^a \wedge e_c \wedge i(e^b) - \frac{1}{2}\sum_{a,b,c} f^c{}_{ab}\, e^a \wedge e^b \wedge i(e_c)\right) - \lambda^2 \sum_{a,b,c} f^c{}_{ab}\, e_c \wedge i(e^a) i(e^b)$$

and similar $\mathrm{ad}_{\mathrm{S}}(\boldsymbol{J}) = \sum_a e^a \wedge \mathrm{ad}_*(\boldsymbol{J}_a) + 2\mathrm{i}\lambda\boldsymbol{M}_{\mathrm{S}}$. Together with (45), Lemma 15, and (12) applied for $\boldsymbol{\delta}$ the theorem follows. $\square$

As we have seen $\boldsymbol{\mathcal{D}}_{\mathrm{S}}$ splits into two super-commuting differentials $\boldsymbol{\delta}$ and $\boldsymbol{\partial}$, a fact which is only visible in this clarity in the standard ordered case. In the general κ-ordered case and in particular in the Weyl case a direct computation of $\boldsymbol{\mathcal{D}}_\kappa$ would be less useful. Hence we transform $\boldsymbol{\delta}$ and $\boldsymbol{\partial}$ back via the equivalence transformation S_κ to find the differentials also in the κ-ordered case. We define

$$\boldsymbol{\delta}_\kappa := S_\kappa^{-1}\boldsymbol{\delta}S_\kappa \quad \text{and} \quad \boldsymbol{\partial}_\kappa := S_\kappa^{-1}\boldsymbol{\partial}S_\kappa. \tag{53}$$

It turns out that neither $\boldsymbol{\delta}_\kappa$ nor $\boldsymbol{\partial}_\kappa$ respect the ghost/antighost degree. To find a more explicit form for $\boldsymbol{\delta}_\kappa$ and $\boldsymbol{\partial}_\kappa$ we need the following lemma:

Lemma 18 *Let $\kappa \in [0,1]$ then $\Delta\boldsymbol{q} - \boldsymbol{q}\Delta = \boldsymbol{c}$, and Δ commutes with $\boldsymbol{c}$, $\boldsymbol{M}_S$, $\boldsymbol{M}_A$, and $\boldsymbol{u}$. Hence $S_\kappa^{-1}\boldsymbol{q}S_\kappa = \boldsymbol{q} - 2\mathrm{i}\kappa\lambda\boldsymbol{c}$, and $S_\kappa^{-1}\boldsymbol{M}_S S_\kappa = \boldsymbol{M}_S$, $S_\kappa^{-1}\boldsymbol{M}_A S_\kappa = \boldsymbol{M}_A$, and $S_\kappa^{-1}\boldsymbol{u}S_\kappa = \boldsymbol{u}$. Finally*

$$\begin{aligned} \Delta\boldsymbol{\delta} - \boldsymbol{\delta}\Delta &= -2\boldsymbol{q} - \tfrac{1}{\mathrm{i}\lambda}(\boldsymbol{M}_A - \boldsymbol{M}_S) + \boldsymbol{u}, \\ S_\kappa^{-1}\boldsymbol{\delta}S_\kappa &= \boldsymbol{\delta} + 4\mathrm{i}\kappa\lambda\boldsymbol{q} + 2\kappa(\boldsymbol{M}_A - \boldsymbol{M}_S) - 2\mathrm{i}\kappa\lambda\boldsymbol{u} + 4\kappa^2\lambda^2\boldsymbol{c}. \end{aligned} \tag{54}$$

PROOF: Using the expressions in terms of a basis for these various operators the above commutation relations follow from a tedious but straightforward computation. $\square$

Using this lemma we can now state the following theorem which gives an explicit form for all κ-ordered BRST operators.

Theorem 19 *Let $\kappa \in [0,1]$ then the κ-ordered BRST operator $\boldsymbol{\mathcal{D}}_\kappa$ splits into two super-commuting differentials $\boldsymbol{\mathcal{D}}_\kappa = \boldsymbol{\delta}_\kappa + 2\boldsymbol{\partial}_\kappa$, where*

$$\boldsymbol{\delta}_\kappa = \boldsymbol{\delta} + 4\mathrm{i}\kappa\lambda\boldsymbol{q} - 2\kappa(\boldsymbol{M}_S - \boldsymbol{M}_A) - 2\mathrm{i}\kappa\lambda\boldsymbol{u} + 4\kappa^2\lambda^2\boldsymbol{c}, \tag{55}$$

$$\boldsymbol{\partial}_\kappa = \boldsymbol{\partial} - 2\kappa\lambda^2\boldsymbol{c} = \boldsymbol{M}_S + \frac{\mathrm{i}\lambda}{2}\boldsymbol{u} - \mathrm{i}\lambda\boldsymbol{q} - 2\kappa\lambda^2\boldsymbol{c}, \tag{56}$$

$$\boldsymbol{\mathcal{D}}_\kappa = \boldsymbol{\delta} + 2\left((1-\kappa)\boldsymbol{M}_S + \kappa\boldsymbol{M}_A\right) + 2\mathrm{i}\lambda(2\kappa-1)\boldsymbol{q} - \mathrm{i}\lambda(2\kappa-1)\boldsymbol{u} - 4\kappa(1-\kappa)\lambda^2\boldsymbol{c}. \tag{57}$$

PROOF: This follows from the very definitions and the last lemma. $\square$

Remark 20 *In the standard ordered case the splitting of $\boldsymbol{\mathcal{D}}_{\mathrm{S}}$ is rather simple to find since in this case the two differentials are homogeneous of bidegree $(1,0)$ resp. $(0,-1)$. Nevertheless for physical reasons one is mostly interested in the Weyl ordered case since only here the pointwise complex conjugation is a super involution (a fact which we shall not need in this paper). But in the Weyl case the splitting is less obvious (and not compatible with the degrees) whence the standard ordered case is a useful tool. In particular we have*

$$\begin{aligned} \boldsymbol{\delta}_{\mathrm{W}} &= \boldsymbol{\delta} + 2\mathrm{i}\lambda\boldsymbol{q} - (\boldsymbol{M}_S - \boldsymbol{M}_A) - \mathrm{i}\lambda\boldsymbol{u} + \lambda^2\boldsymbol{c}, \\ \boldsymbol{\partial}_{\mathrm{W}} &= \boldsymbol{M}_S + \tfrac{\mathrm{i}\lambda}{2}\boldsymbol{u} - \mathrm{i}\lambda\boldsymbol{q} - \lambda^2\boldsymbol{c}, \\ \boldsymbol{\mathcal{D}}_{\mathrm{W}} &= \boldsymbol{\delta} + \boldsymbol{M}_S + \boldsymbol{M}_A - \lambda^2\boldsymbol{c}. \end{aligned} \tag{58}$$

We shall denote the cohomology $\ker \boldsymbol{\mathcal{D}}_{\mathrm{S}} \big/ \operatorname{im} \boldsymbol{\mathcal{D}}_{\mathrm{S}}$ of the quantised standard ordered BRST differential by $\boldsymbol{H}^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\lambda]])$ and shall speak of the *quantum BRST cohomology.* The cohomology of $\boldsymbol{\delta}$ will be denoted by $\boldsymbol{H}^{\bullet}_{\mathrm{CE}}(\mathcal{A}[[\lambda]])$, and the homology of $\boldsymbol{\partial}$ by $\boldsymbol{H}^{\mathrm{Kos}}_{\bullet}(\mathcal{A}[[\lambda]])$. Exactly as in the classical case we get the following quantum analogue of Lemma 9:

Lemma 21 *The quantum BRST cohomology is equipped with a natural $\mathbb{Z}$-graded associative $\mathbb{C}[[\lambda]]$-bilinear multiplication $\star_{\mathrm{S}}$ induced by the associative multiplication $\star_{\mathrm{S}}$ of the quantum BRST algebra $\mathcal{A}[[\lambda]]$: let $a, b \in \mathcal{A}$ such that $\boldsymbol{\mathcal{D}}_{\mathrm{S}}a = 0 = \boldsymbol{\mathcal{D}}_{\mathrm{S}}b$; then for the corresponding cohomology classes $[a], [b] \in H^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\lambda]])$ we have $[a] \star_{\mathrm{S}} [b] := [a \star_{\mathrm{S}} b]$.*

Moreover, the equivalence transformation S_κ renders all the cohomologies of the κ-ordered BRST differentials canonically isomorphic as associative algebras for all $\kappa \in [0,1]$.

The proof is completely analogous to the proof of Lemma 9.

We shall now introduce a quantum analogue of the classical ideal generated by J, $\mathcal{I}(J)$ (see Section 3 for a definition) and its normalizer thereby generalising a notion also used by Glößner in [25]:

Definition 22 *i.) Let $\boldsymbol{\mathcal{I}}(\boldsymbol{J}) := \boldsymbol{\partial}(\mathcal{A}^{0,1}[[\lambda]]) \subset \mathcal{A}^{0,0}[[\lambda]]$ be the quantum ideal of $\boldsymbol{J}$.*

ii.) Let $\boldsymbol{\mathcal{B}}(\boldsymbol{J}) := \left\{ f \in C^\infty(M)[[\lambda]] \;\middle|\; f * g - g * f \in \boldsymbol{\mathcal{I}}(\boldsymbol{J}) \quad \forall g \in \boldsymbol{\mathcal{I}}(\boldsymbol{J}) \right\}$ denote the quantum idealiser of $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$.

Proposition 23 *i.) $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ is a left ideal of the algebra $(C^\infty(M)[[\lambda]], *)$. Moreover $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ is stable under the representation $\boldsymbol{\varrho}_M$, see (4).*

ii.) $\boldsymbol{\mathcal{B}}(\boldsymbol{J})$ *is a sub-algebra of the algebra* $(C^\infty(M)[[\lambda]], *)$ *containing* $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ *as a two-sided ideal.*

iii.) The quotient $\boldsymbol{\mathcal{B}}(\boldsymbol{J})/\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ *becomes an associative algebra in a canonical way.*

PROOF: First note that the operator $\boldsymbol{q}$ vanishes on $\mathcal{A}^{0,1}[[\lambda]]$, hence every element of $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ is equal to a sum of elements of the form $a*(\langle \boldsymbol{J}+\lambda\chi, \xi\rangle)$ where $a \in C^\infty(M)[[\lambda]]$ and $\xi \in \mathfrak{g}$ which immediately shows that $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ is indeed a left ideal of the algebra $C^\infty(M)[[\lambda]]$. Furthermore, $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ is stable under the above representation thanks to quantum covariance and the fact that χ vanishes on commutators. The second part is true by abstract algebra and follows from associativity and the third part is clear. $\square$

Note that $\boldsymbol{\mathcal{B}}(\boldsymbol{J})$ is the idealiser of the left ideal $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ in the sense of [30, Eq. (21), p. 199]. In physics the ideal $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$ may roughly be interpreted as the space of all those 'operators in the big unphysical Hilbert space' vanishing on the smaller 'physical Hilbert space', whereas $\boldsymbol{\mathcal{B}}(\boldsymbol{J})$ will then be the space of all those 'operators in the big unphysical Hilbert space' leaving invariant the 'physical subspace' whence the quotient algebra serves as the observable algebra on the physical Hilbert space.

At the end of this section we should like to mention an important algebra homomorphism relating the ghost-number zero part of the quantum BRST cohomology to the above quotient algebra in Proposition 23:

Proposition 24 *Let* $(M, *, \mathfrak{g}, \boldsymbol{J})$ *be a Hamiltonian quantum* $\mathfrak{g}$*-space. Then there is a canonical homomorphism* L *of associative algebras*

$$L : \boldsymbol{H}^0_{\mathrm{BRST}}(\mathcal{A}[[\lambda]]) \to \boldsymbol{\mathcal{B}}(\boldsymbol{J})/\boldsymbol{\mathcal{I}}(\boldsymbol{J}), \tag{59}$$

mapping each point $[\phi = \sum_{i=0}^n \phi_i] \in \boldsymbol{H}^0_{\mathrm{BRST}}(\mathcal{A}[[\lambda]])$ *where* $\phi_i \in \mathcal{A}^{i,i}$ *and* $\boldsymbol{\mathcal{D}}_{\mathrm{S}}\phi = 0$ *to* $\phi_0 \bmod \boldsymbol{\mathcal{I}}(\boldsymbol{J})$.

PROOF: Note that the equation $\boldsymbol{\mathcal{D}}_{\mathrm{S}}\phi = 0$ means in lowest order that for ϕ_0 there is ϕ_1 such that $\boldsymbol{\delta}\phi_0 + 2\boldsymbol{\partial}_1\phi_1 = 0$. Evaluating on $\mathfrak{g}$ this means that $\langle \boldsymbol{J}, \xi\rangle * \phi_0 - \phi_0 * \langle \boldsymbol{J}, \xi\rangle$ is contained in $\boldsymbol{\mathcal{I}}(\boldsymbol{J})$. Since $\chi(\xi)$ obviously commutes with every ϕ_0 and using associativity we see that ϕ_0 has to be in $\boldsymbol{\mathcal{B}}(\boldsymbol{J})$. Moreover in case $\phi = \boldsymbol{\mathcal{D}}_{\mathrm{S}}\phi'$ this means that $\phi_0 = \boldsymbol{\partial}_1\phi'_0$ where $\phi'_0 \in \mathfrak{g} \otimes C^\infty(M)[[\lambda]]$ whence the BRST quantum coboundaries are mapped to the quantum ideal of $\boldsymbol{J}$ which shows that the map L is well-defined. In order to see that L is a homomorphism of associative algebras one only has to observe that the component in $\mathcal{A}^{0,0}$ of the multiplication $\phi \star_{\mathrm{S}} \psi$ (where again $\psi \in \mathcal{A}^{(0)}[[\lambda]]$ with $\psi = \sum_{i=0}^n \psi_i$, $\psi_i \in \mathcal{A}^{i,i}$ and $\boldsymbol{\mathcal{D}}_{\mathrm{S}}\psi = 0$) is simply equal to $\phi_0 * \psi_0$ thanks to the fact that

$$\mathcal{A}^{i,j}[[\lambda]] \star_{\mathrm{S}} \mathcal{A}^{k,l}[[\lambda]] \subset \sum_{r=0}^{\min\{j,k\}} \mathcal{A}^{i+k-r, j+l-r}[[\lambda]]$$

which is particular for the standard ordered product $\star_{\mathrm{S}}$ on $\mathcal{A}[[\lambda]]$. $\square$

6 Computation of the quantum BRST Cohomology

In Theorem 19 we have shown that the standard ordered quantum BRST operator forms a double complex. As already mentioned both quantised differentials $\boldsymbol{\delta}$, $\boldsymbol{\partial}$ are deformations of the corresponding classical operators δ, ∂. This enables us to simplify the quantum BRST cohomology in exactly the same way as it was done for the classical cohomology in Section 4.

In order to get the full quantum analogue of Lemma 5 and of Proposition 10 we still have to 'quantise' the augmentation, i.e. to define a reasonable deformation of the restriction map ι^*. In the following proposition all the maps will be defined on the whole quantum BRST algebra. Moreover, we shall suppose from now on that (M, ω) is symplectic and $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ is a Hamiltonian quantum $\mathfrak{g}$-space with regular constraint surface, and we shall write $\boldsymbol{\mathcal{I}}_C$ for the quantum ideal generated by $\boldsymbol{J}$ now speaking of it as the *quantum vanishing ideal of* C. Likewise, its quantum idealiser $\boldsymbol{\mathcal{B}}(\boldsymbol{J})$ will now be denoted by $\boldsymbol{\mathcal{B}}_C$ to get an analogy to the classical case (see Lemma 4).

Proposition 25 *Using the notation of Lemma 5 and of the two preceding sections we define* $B_1 := \frac{1}{\lambda}(\partial_1 - \boldsymbol{\partial}_1)$ *and, using the chain homotopy* h_0 *(see Lemma 5), the deformed restriction map*

$$\boldsymbol{\iota}^* := \iota^*(\mathsf{id}_0 - \lambda B_1 h_0)^{-1} = \sum_{r=0}^{\infty} \lambda^r \boldsymbol{\iota}^*{}_r, \tag{60}$$

where we have written $\mathsf{id}_0 := \mathsf{id}_{\bigwedge \mathfrak{g}^* \otimes C^\infty(M)[[\lambda]]}$. *Then* $\boldsymbol{\iota}^*$ *is the unique* $\mathbb{C}[[\lambda]]$*-linear map* $\bigwedge \mathfrak{g}^* \otimes C^\infty(M)[[\lambda]] \to \bigwedge \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]]$ *which satisfies*

$$\begin{aligned} \boldsymbol{\iota}^*{}_0 &= \iota^*, & (61)\\ \boldsymbol{\iota}^* \boldsymbol{\partial}_1 &= 0, & (62)\\ \boldsymbol{\iota}^* \operatorname{prol} &= \mathsf{id}_{-1}, & (63) \end{aligned}$$

where we have written $\mathsf{id}_{-1} := \mathsf{id}_{\bigwedge \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]]}$. *Moreover the map* $\operatorname{prol} \boldsymbol{\iota}^*$ *is a projection onto* $\bigwedge \mathfrak{g}^* \otimes \mathcal{F}_C[[\lambda]]$ *whose kernel is given by* $\bigwedge \mathfrak{g}^* \otimes \boldsymbol{\mathcal{I}}_C$. *In particular, we have the decomposition*

$$\bigwedge \mathfrak{g}^* \otimes C^\infty(M)[[\lambda]] = (\bigwedge \mathfrak{g}^* \otimes \mathcal{F}_C)[[\lambda]] \oplus (\bigwedge \mathfrak{g}^* \otimes \boldsymbol{\mathcal{I}}_C). \tag{64}$$

PROOF: Since $\boldsymbol{\partial}$ is a deformation of ∂ the formal series of differential operators B_1 is well-defined. Moreover, we have $\mathsf{id}_0 = \operatorname{prol} \iota^* + \partial_1 h_0 = \operatorname{prol} \iota^* + \boldsymbol{\partial}_1 h_0 + \lambda B_1 h_0$ whence

$$\mathsf{id}_0 = \operatorname{prol} \boldsymbol{\iota}^* + \boldsymbol{\partial}_1 h_0 (\mathsf{id}_0 - \lambda B_1 h_0)^{-1}. \tag{65}$$

Because $h_0 \operatorname{prol} = 0$ (Eq. (9)) we immediately see that $\boldsymbol{\iota}^* \operatorname{prol} = \mathsf{id}_{-1}$ (since $\iota^* \operatorname{prol} = \mathsf{id}_{-1}$) which implies that $\operatorname{prol} \boldsymbol{\iota}^*$ is a projection. This fact together with (65) entails that the product $\operatorname{prol} \boldsymbol{\iota}^* \boldsymbol{\partial}_1 h_0 (\mathsf{id}_0 - \lambda B_1 h_0)^{-1}$ vanishes, and since prol is injective and $(\mathsf{id}_0 - \lambda B_1 h_0)^{-1}$ is invertible we get $\boldsymbol{\iota}^* \boldsymbol{\partial}_1 h_0 = 0$. Multiplying $\boldsymbol{\partial}_1 h_0$ by ∂_1 from the right we obtain the following, using Lemma 5 and writing id_1 for $\mathsf{id}_{\bigwedge \mathfrak{g}^* \wedge \mathfrak{g} \otimes C^\infty(M)[[\lambda]]}$

$$\boldsymbol{\partial}_1 h_0 \partial_1 = \boldsymbol{\partial}_1 (\mathsf{id}_1 - \partial_2 h_1) = \boldsymbol{\partial}_1 (\mathsf{id}_1 - (\partial_2 - \boldsymbol{\partial}_2) h_1), \tag{66}$$

thanks to $0 = \boldsymbol{\partial}_1 \boldsymbol{\partial}_2$. But the difference $\partial_2 - \boldsymbol{\partial}_2$ is a multiple of λ whence the right factor of the second equation above is invertible which immediately implies $\boldsymbol{\iota}^* \boldsymbol{\partial}_1 = 0$. The same argument (66) also shows that the image of $\boldsymbol{\partial}_1$ is equal to the image of the projector $\boldsymbol{\partial}_1 h_0 (\mathsf{id}_0 - \lambda B_1 h_0)^{-1}$. It follows that $\bigwedge \mathfrak{g}^* \otimes \boldsymbol{\mathcal{I}}_C$ is the kernel of the projection $\operatorname{prol} \boldsymbol{\iota}^*$. Moreover, let $\phi \in C^\infty(C)[[\lambda]]$. Then $\operatorname{prol} \boldsymbol{\iota}^* \operatorname{prol}(\phi) = \operatorname{prol}(\phi)$ which shows that the image of $\operatorname{prol} \boldsymbol{\iota}^*$ is given by $\mathcal{F}_C[[\lambda]]$. This proves the direct sum decomposition after tensoring with the Graßmann algebra. Now suppose that there were another such deformed restriction map, $\boldsymbol{\iota}^{*\prime}$, satisfying the conditions of the Proposition. Then (63) implies that $\operatorname{prol} \boldsymbol{\iota}^{*\prime}$ is a projection whose image is $\bigwedge \mathfrak{g}^* \otimes \mathcal{F}_C[[\lambda]]$. Moreover, (62) entails that the image of $\boldsymbol{\partial}_1$, $\boldsymbol{\mathcal{I}}_C$, is contained in the kernel of this projection. Thanks to the already proven direct sum decomposition (64) the kernel of the projection $\operatorname{prol} \boldsymbol{\iota}^{*\prime}$ decomposes into the direct sum of the Graßmann algebra tensor the quantum vanishing ideal and the intersection of this kernel with $\bigwedge \mathfrak{g}^* \otimes \mathcal{F}_C[[\lambda]]$: this is impossible by (63). Hence $\operatorname{prol}(\boldsymbol{\iota}^* - \boldsymbol{\iota}^{*\prime}) = 0$ which implies equality of the two quantum restriction maps by injectivity of prol. $\square$

We shall now define the augmented quantum BRST complex and the deformed chain homotopies in a manner analogous to the classical case: let $\hat{\mathcal{A}}[[\lambda]]$ denote the *augmented BRST complex* $(\bigwedge \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]]) \oplus \mathcal{A}[[\lambda]]$. Let $\hat{\boldsymbol{\partial}} : \hat{\mathcal{A}}[[\lambda]] \to \hat{\mathcal{A}}[[\lambda]]$ be equal to the quantised Koszul operator $\boldsymbol{\partial}$ on the quantum BRST complex $\mathcal{A}^{\bullet,>0}[[\lambda]]$, equal to $\boldsymbol{\iota}^*$ on $\mathcal{A}^{\bullet,0}[[\lambda]]$, and zero otherwise. The deformed chain homotopies on the augmented quantised BRST complex will be defined in the following

Proposition 26 *Define the quantised chain homotopy* $\hat{\boldsymbol{h}} : \hat{\mathcal{A}}[[\lambda]] \to \hat{\mathcal{A}}[[\lambda]]$ *in the following way: on the subspace* $\bigwedge \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]]$ *set it equal to the classical prolongation, i.e.* $\hat{\boldsymbol{h}}_{-1} := \operatorname{prol}$ *and on the quantum BRST complex* $\mathcal{A}[[\lambda]]$ *set* $\hat{\boldsymbol{h}} := \boldsymbol{h}$ *where the latter is given as*

$$\begin{aligned} \boldsymbol{h}_0 &:= h_0(\operatorname{prol} \boldsymbol{\iota}^* + \boldsymbol{\partial}_1 h_0)^{-1} = h_0(\mathsf{id}_0 - \lambda B_1 h_0)^{-1}, & (67)\\ \boldsymbol{h}_i &:= h_i(h_{i-1} \boldsymbol{\partial}_i + \boldsymbol{\partial}_{i+1} h_i)^{-1} \qquad \forall i \geq 1, & (68) \end{aligned}$$

where B_1 is as in Proposition 25. Furthermore we get the chain homotopy equation on the augmented quantum BRST complex

$$\hat{\boldsymbol{\partial}}\hat{\boldsymbol{h}} + \hat{\boldsymbol{h}}\hat{\boldsymbol{\partial}} = \mathsf{id}, \tag{69}$$

whence the quantised augmented Koszul complex also has trivial homology.

PROOF: Firstly, note that the fact that $\hat{\boldsymbol{\partial}}$ is a deformation of $\hat{\partial}$ implies that $\hat{\boldsymbol{h}}$ is a well-defined deformation of the classical augmented chain homotopy $\hat{h}$. Since $\hat{\boldsymbol{\partial}}^2 = 0$ it follows that $(\hat{h}\hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\partial}}\hat{h})$ commutes with $\hat{\boldsymbol{\partial}}$ which immediately implies (69). Moreover, since the restriction of $(\hat{h}\hat{\boldsymbol{\partial}} + \hat{\boldsymbol{\partial}}\hat{h})$ to $C^\infty(M)[[\lambda]]$ is equal to

$$\mathrm{prol}\,\boldsymbol{\iota}^* + \boldsymbol{\partial}_1 h_0 = \mathsf{id}_0 - \lambda(\mathsf{id}_0 - \mathrm{prol}\,\boldsymbol{\iota}^*)B_1 h_0,$$

we see that the terms containing the projection $\mathrm{prol}\,\boldsymbol{\iota}^*$ in (67) vanish thanks to $h_0\,\mathrm{prol} = 0$, see (9), which shows the second equation in (67). □

The following lemma is a key to questions of bidifferentiability of the reduced star product:

Lemma 27 *Using the geometric chain homotopy h_0 of Lemma 5 there is a formal series of differential linear operators of $C^\infty(M)$, $S := \mathsf{id}_{C^\infty(M)} + \sum_{r=1}^\infty \lambda^r S_r$ where S_r vanishes on constants for $r \geq 1$, such that*

$$\boldsymbol{\iota}^* = \iota^* \circ S.$$

In particular, if $*$ is of Vey type then the order of each S_r is at most r. Moreover, under the additional assumptions of Lemma 3 S can be chosen to be G-invariant.

PROOF: It suffices to show that for any differential operator $D : \mathfrak{g} \otimes C^\infty(M) \to C^\infty(M)$ of order k there is a differential operator $D' : C^\infty(M) \to C^\infty(M)$ of order $k+1$ such that the following holds in the tubular neighbourhood of Lemma 5:

$$\iota^* \circ D \circ h_0 = \iota^* \circ D'.$$

But this is clear: the tubular neighbourhood is diffeomorphic to an open subset of $C \times \mathfrak{g}^*$, hence using the classical momentum map J as a global coordinate we see that D takes the form

$$D = \sum_{s=0}^{k} \sum_{j,j_1,\ldots,j_s=1}^{\dim\mathfrak{g}} i(e^j) {D^{(s)}}_{jj_1\cdots j_s} \frac{\partial^s}{\partial J_{j_1}\cdots\partial J_{j_s}},$$

where ${D^{(s)}}_{jj_1\cdots j_s}$ are differential operators $C^\infty(C) \to C^\infty(C)$ of order $k-s$ which are smoothly parametrised by $J_1, \ldots, J_n$, and where $e^1, \ldots, e^n$ is a basis of $\mathfrak{g}^*$. Using formula (10) for h_0 near C we see that

$$D' = \sum_{s=0}^{k} \sum_{j,j_1,\ldots,j_s=1}^{\dim\mathfrak{g}} \frac{1}{s+1} {D^{(s)}}_{jj_1\cdots j_s} \frac{\partial^{s+1}}{\partial J_j \partial J_{j_1}\cdots\partial J_{j_s}}$$

will satisfy the above equation and is clearly G-invariant if D is G-equivariant. By induction, the composition of the restriction ι^* and finitely many powers of $\lambda B_1 h_0$ will be of the desired differential operator form with the correct bounds for the orders. This will define the operators S_r, $r \geq 1$, of the asserted formula in the (G-invariant) tubular neighbourhood. Since by the very definition of a star product B_1 vanishes on the constant functions, the S_r will also vanish on the constants for $r \geq 1$. Multiplying the S_r, $r \geq 1$, by a suitable (G-invariant) bump function with support in the tubular neighbourhood S (and equal to 1 in an open neighbourhood of C) will give us a globally defined operator series S still satisfying the asserted equation. □

We shall now need the quantum analogue of the Lie algebra representation ϱ_C on the constraint surface to construct the quantum analogue of the Chevalley-Eilenberg differential δ^c. The motivating classical equation is the identity (23), and we set for all $\xi \in \mathfrak{g}$

$$\boldsymbol{\varrho}_C(\xi) := \boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\xi)\,\mathrm{prol}\,. \tag{70}$$

where the representation $\boldsymbol{\varrho}_M$ is defined in (4).

Lemma 28 *The map $\boldsymbol{\varrho}_C$ defines a Lie algebra representation of $\mathfrak{g}$ on the space $C^\infty(C)[[\lambda]]$. Moreover, the quantised restriction map $\boldsymbol{\iota}^*$ induces a $\mathfrak{g}$-module isomorphism of the $\mathfrak{g}$-module $C^\infty(M)[[\lambda]]/\boldsymbol{\mathcal{I}}_C$ (compare Lemma 23) onto the $\mathfrak{g}$-module $C^\infty(C)[[\lambda]]$. More precisely, we have the following identity for all $\xi \in \mathfrak{g}$:*

$$\boldsymbol{\varrho}_C(\xi)\boldsymbol{\iota}^* = \boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\xi). \tag{71}$$

Proof: Since the map $\mathrm{prol}\,\boldsymbol{\iota}^*$ is a projection (see Proposition 25) whose kernel (after restriction to ghost-number zero) is equal to the quantum vanishing ideal it follows from Lemma 23 that for all $\xi \in \mathfrak{g}$

$$\mathrm{prol}\,\boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\xi)\,\mathrm{prol}\,\boldsymbol{\iota}^* = \mathrm{prol}\,\boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\xi).$$

Hence for all $\xi, \eta \in \mathfrak{g}$

$$\mathrm{prol}\,\boldsymbol{\varrho}_C(\xi)\boldsymbol{\varrho}_C(\eta)\boldsymbol{\iota}^* = \mathrm{prol}\,\boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\xi)\,\mathrm{prol}\,\boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\eta)\,\mathrm{prol}\,\boldsymbol{\iota}^* = \mathrm{prol}\,\boldsymbol{\iota}^* \boldsymbol{\varrho}_M(\xi)\boldsymbol{\varrho}_M(\eta)\,\mathrm{prol}\,\boldsymbol{\iota}^*,$$

whence the representation identity follows since prol is injective and $\boldsymbol{\iota}^*$ is surjective. The rest of the lemma is clear thanks to Proposition 25. □

As in the classical case (22) there is the following simple consequence for the corresponding quantum Chevalley-Eilenberg differential, $\boldsymbol{\delta}^c$, (see (11) for a definition) on the constraint surface:

$$\boldsymbol{\delta}^c \boldsymbol{\iota}^* = -\boldsymbol{\iota}^* \boldsymbol{\delta}. \tag{72}$$

In complete analogy to the classical case we denote by $\hat{\boldsymbol{\mathcal{D}}}_{\mathrm{S}}$ the augmented quantum $BRST$ operator

$$\hat{\boldsymbol{\mathcal{D}}}_{\mathrm{S}} := \boldsymbol{\delta}^c + 2\boldsymbol{\iota}^* + \boldsymbol{\mathcal{D}}_{\mathrm{S}}, \tag{73}$$

where all the maps are defined to be zero on the domains on which they were previously not defined. Clearly

$$\hat{\boldsymbol{\mathcal{D}}}_{\mathrm{S}}^{\,2} = 0. \tag{74}$$

The augmented quantum BRST complex is depicted in Figure 1. We keep the notation $\hat{\boldsymbol{h}}$ for $\mathrm{prol} + \boldsymbol{h}$ on $\hat{\mathcal{A}}$. Moreover, let $\hat{\boldsymbol{\delta}}$ denote the augmented Chevalley-Eilenberg operator $\boldsymbol{\delta}^c + \boldsymbol{\delta}$.

We are now going to compute the cohomology of the BRST complex in terms of the quantum Chevalley-Eilenberg cohomology on the constraint surface:

Theorem 29 *With the above notations and definitions we have the following:*

i.) The following map

$$\hat{\boldsymbol{h}}' := \frac{1}{2}\hat{\boldsymbol{h}}\big(\mathsf{id} + \frac{1}{2}(\hat{\boldsymbol{\delta}}\hat{\boldsymbol{h}} + \hat{\boldsymbol{h}}\hat{\boldsymbol{\delta}})\big)^{-1} \tag{75}$$

is a homotopy for the differential $\hat{\boldsymbol{\mathcal{D}}}_{\mathrm{S}}$, i.e. $\hat{\boldsymbol{\mathcal{D}}}_{\mathrm{S}}\hat{\boldsymbol{h}}' + \hat{\boldsymbol{h}}'\hat{\boldsymbol{\mathcal{D}}}_{\mathrm{S}} = \mathsf{id}$. The $\mathbb{C}[[\lambda]]$-linear map

$$\boldsymbol{\Psi} : \boldsymbol{H}^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\lambda]]) \to \boldsymbol{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)[[\lambda]]) : \quad [a] \mapsto [\boldsymbol{\iota}^* a] \tag{76}$$

is an isomorphism with the following inverse (where $[c] \in \boldsymbol{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)[[\lambda]])$):

$$\boldsymbol{\Psi}^{-1} : [c] \mapsto \left[2\hat{\boldsymbol{h}}' c\right] = \sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k \left[(\boldsymbol{h}\boldsymbol{\delta})^k\,\mathrm{prol}\,c\right]. \tag{77}$$

$$
\begin{tikzcd}
\bigwedge^n \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]] & \mathcal{A}^{n,0}[[\lambda]] \arrow[l, "\boldsymbol{\iota}^*"'] & \mathcal{A}^{n,1}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_1"'] & \cdots\cdots \arrow[l, "\boldsymbol{\partial}_2"'] & \mathcal{A}^{n,n-1}[[\lambda]] \arrow[l] & \mathcal{A}^{n,n}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_n"'] \\
\bigwedge^{n-1} \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]] \arrow[u, "\boldsymbol{\delta}^c"] & \mathcal{A}^{n-1,0}[[\lambda]] \arrow[l, "\boldsymbol{\iota}^*"'] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{n-1,1}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_1"'] \arrow[u, "\boldsymbol{\delta}"] & \cdots\cdots \arrow[l, "\boldsymbol{\partial}_2"'] & \mathcal{A}^{n-1,n-1}[[\lambda]] \arrow[l] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{n-1,n}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_n"'] \arrow[u, "\boldsymbol{\delta}"] \\
\bigwedge^{n-2} \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]] \arrow[u, "\boldsymbol{\delta}^c"] & \mathcal{A}^{n-2,0}[[\lambda]] \arrow[l, "\boldsymbol{\iota}^*"'] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{n-2,1}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_1"'] \arrow[u, "\boldsymbol{\delta}"] & \cdots\cdots \arrow[l, "\boldsymbol{\partial}_2"'] & \mathcal{A}^{n-2,n-1}[[\lambda]] \arrow[l] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{n-2,n}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_n"'] \arrow[u, "\boldsymbol{\delta}"] \\
{} \arrow[u] & \cdots\cdots \arrow[u] & \cdots\cdots \arrow[u] & \cdots\cdots & {} \arrow[u] & \cdots\cdots \arrow[u] \\
\bigwedge^2 \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]] \arrow[u] & \mathcal{A}^{2,0}[[\lambda]] \arrow[l, "\boldsymbol{\iota}^*"'] \arrow[u] & \mathcal{A}^{2,1}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_1"'] \arrow[u] & \cdots\cdots \arrow[l, "\boldsymbol{\partial}_2"'] & \mathcal{A}^{2,n-1}[[\lambda]] \arrow[l] \arrow[u] & \mathcal{A}^{2,n}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_n"'] \arrow[u] \\
\mathfrak{g}^* \otimes C^\infty(C)[[\lambda]] \arrow[u, "\boldsymbol{\delta}^c"] & \mathcal{A}^{1,0}[[\lambda]] \arrow[l, "\boldsymbol{\iota}^*"'] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{1,1}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_1"'] \arrow[u, "\boldsymbol{\delta}"] & \cdots\cdots \arrow[l, "\boldsymbol{\partial}_2"'] & \mathcal{A}^{1,n-1}[[\lambda]] \arrow[l] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{1,n}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_n"'] \arrow[u, "\boldsymbol{\delta}"] \\
C^\infty(C)[[\lambda]] \arrow[u, "\boldsymbol{\delta}^c"] & \mathcal{A}^{0,0}[[\lambda]] \arrow[l, "\boldsymbol{\iota}^*"'] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{0,1}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_1"'] \arrow[u, "\boldsymbol{\delta}"] & \cdots\cdots \arrow[l, "\boldsymbol{\partial}_2"'] & \mathcal{A}^{0,n-1}[[\lambda]] \arrow[l] \arrow[u, "\boldsymbol{\delta}"] & \mathcal{A}^{0,n}[[\lambda]] \arrow[l, "\boldsymbol{\partial}_n"'] \arrow[u, "\boldsymbol{\delta}"] \\
C^\infty(M_{\mathrm{red}})[[\lambda]] \arrow[u, "\pi^*"] & & & & &
\end{tikzcd}
$$

Figure 1: The augmented quantum BRST complex.

ii.) The isomorphism $\boldsymbol{\Psi}$ turns the quantum Chevalley-Eilenberg cohomology on the constraint surface, $\boldsymbol{H}^\bullet_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)[[\lambda]])$, into a $\mathbb{Z}$-graded associative algebra with unit. The multiplication (also called star product) $*$ for $c_1, c_2 \in \bigwedge \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]]$ takes the following form:

$$
[c_1] * [c_2] := 4\left[\boldsymbol{\iota}^*\left((\hat{\boldsymbol{h}}' c_1) \star_{\mathrm{S}} (\hat{\boldsymbol{h}}' c_2)\right)\right] \tag{78}
$$

iii.) The homomorphism $L : \boldsymbol{H}^0_{\mathrm{BRST}}(\mathcal{A}[[\lambda]]) \to \boldsymbol{\mathcal{B}}_C/\boldsymbol{\mathcal{I}}_C$ in Proposition 24 is an isomorphism of associative algebras.

iv.) For $c_1, c_2 \in \boldsymbol{H}^0_{\mathrm{BRST}}(\mathcal{A}[[\lambda]]) = (C^\infty(C)[[\lambda]])^{\mathfrak{g}}$ formula (78) simplifies to

$$
c_1 * c_2 = \boldsymbol{\iota}^*\left((\operatorname{prol} c_1) \star_{\mathrm{S}} (\operatorname{prol} c_2)\right). \tag{79}
$$

PROOF: The proof is entirely analogous to the proof of the classical Proposition 10 since only purely cohomological statements are needed. For the last two parts note that $\operatorname{prol} c$ is in $\boldsymbol{\mathcal{B}}_C$ if $\boldsymbol{\delta}^c c = 0$ and use the proof of Proposition 24. ☐

Note that, as also in the classical case, one has by pure diagram chase and the acyclicity of the quantum Koszul complex $\boldsymbol{H}^\bullet_{\mathrm{CE}}(\boldsymbol{H}^{\mathrm{Kos}}_0(\mathcal{A}[[\lambda]])) \cong \boldsymbol{H}^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\lambda]])$, but we do not need this fact.

We have thus shown that the quantum BRST cohomology may be computed in terms of the quantum Chevalley-Eilenberg cohomology on the constraint surface. The natural question arises whether this latter cohomology is related to a deformation of functions on the classical reduced phase, at least at level ghost number zero. In the next section we shall see in a simple counter example that this will in general *not* be the case: there are in general 'fewer' quantum invariants than classical invariants. But there will be a large class of positive situations in Section 8. In order to give a precise notion of 'fewer invariants' we shall need the following general definition of a *deformation of a subspace*:

Definition 30 *Let $E : W \hookrightarrow V$ be a subspace of the vector space V over some field k. A $k[[\lambda]]$-submodule $\boldsymbol{W}$ of $V[[\lambda]]$ is called a deformation of W if there exists a deformation of the canonical embedding E i.e. a formal power series of linear maps*

$$\boldsymbol{E} = \boldsymbol{E}_0 + \sum_{r=1}^{\infty} \lambda^r \boldsymbol{E}_r : W[[\lambda]] \to V[[\lambda]] \tag{80}$$

with $\boldsymbol{E}_0 = E$, such that $\boldsymbol{E}(W[[\lambda]]) = \boldsymbol{W}$.

Note that it is not hard to see that in algebraic language the above deformations exactly correspond to those λ-adically closed primary submodules of $V[[\lambda]]$ whose associated prime ideal vanishes (see e.g. [30, p. 434] for a definition) or satisfy the equivalent and more practical condition that $\lambda v \in \boldsymbol{W}$ implies $v \in \boldsymbol{W}$ for all $v \in V[[\lambda]]$, but we shall not need this in the sequel.

We now give a precise definition of a *consistent quantum reduction*:

Definition 31 *Let $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ be a Hamiltonian quantum $\mathfrak{g}$-space regular constraint surface and let $(M, \omega, \mathfrak{g}, J, C)$ be the corresponding classical limit. We shall say that $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ admits a consistent quantum reduction if and only if $\boldsymbol{H}^0_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)[[\lambda]])$ is a deformation of $C^\infty(C)^{\mathfrak{g}}[[\lambda]]$.*

Note that the quantum vanishing ideal $\boldsymbol{\mathcal{I}}_C$ is a deformation of the classical vanishing ideal defining $\boldsymbol{E}$ by the restriction of $\mathsf{id} - \mathrm{prol}\, \boldsymbol{\iota}^*$ to the classical vanishing ideal (where Eqn. (9) is used). However, $\boldsymbol{\mathcal{B}}_C$ is not necessarily a deformation of $\mathcal{B}_C$.

7 An example and counter example

This section is divided in three parts: first, we give an example of a classical phase space reduction (with proper group action and regular values of the momentum) for $M := T^*S^1 \times T^*S^1$. Then we shall show how our method for quantum reduction works for a strongly invariant star product. Finally, we give an explicit example of a non strongly invariant star product which cannot be reduced.

We consider the phase space $M = T^*S^1 \times T^*S^1$. A point in M is described by the quadruple (z, p, w, J) where $z = e^{\mathrm{i}\varphi}$ and $w = e^{\mathrm{i}\psi}$ are the coordinates along S^1 and p resp. J are the corresponding momentum variables. The symplectic form is then given by $\omega = -d\varphi \wedge dp + d\psi \wedge dJ$ where $d\varphi$ resp. $d\psi$ are the non-exact global one-forms on the corresponding circle. As group action we take the $U(1)$-action induced by the coordinate function J which thus serves as momentum map. The constraint surface $C = J^{-1}(\{0\})$ is simply given by $T^*S^1 \times S^1$. On C we use the coordinates (z, p, w), then C is embedded in M by $\iota : (z, p, w) \mapsto (z, p, w, 0)$. Note that the group action is clearly proper and free. Finally, the reduced phase space M_{red} is given by T^*S^1 with coordinates (z, p) and the projection $\pi : C \to M_{\mathrm{red}}$ is just $(z, p, w) \mapsto (z, p)$. Since the Lie algebra $\mathfrak{u}(1)$ is one-dimensional we can identify $\mathfrak{u}(1) \otimes C^\infty(M)$ with $C^\infty(M)$ and similar also $\mathfrak{u}^*(1) \wedge \mathfrak{u}(1) \otimes C^\infty(M)$ and $\mathfrak{u}^*(1) \otimes C^\infty(M)$. Then the classical Koszul operator ∂_1 is just the multiplication with J, i.e.

$$(\partial_1 f)(z, p, w, J) = J f(z, p, w, J) \tag{81}$$

for $f \in C^\infty(M)$. As prolongation we simply use

$$(\mathrm{prol}\, u)(z, p, w, J) = u(z, p, w), \tag{82}$$

where $u \in C^\infty(C)$. Due to the requirement $\mathsf{id} = \mathrm{prol}\, \iota^* + \partial_1 h_0$ the chain homotopy h_0 is given by the difference quotient

$$(h_0 f)(z, p, w, J) = \frac{f(z, p, w, J) - f(z, p, w, 0)}{J} \tag{83}$$

for $J \neq 0$, which is smoothly extended by the first partial derivative in the J-direction on C. Then indeed $\mathsf{id} = \operatorname{prol}\iota^* + \partial_1 h_0$. Finally the classical Lie algebra action on $C^\infty(M)$ is given by the Poisson bracket with J which can be computed explicitly,

$$\varrho_M f = \{J, f\} = -\frac{\partial}{\partial\psi} f, \tag{84}$$

where $f \in C^\infty(M)$. The action on $C^\infty(C)$ is similarly given by

$$\varrho_C u = -\frac{\partial}{\partial\psi} u, \tag{85}$$

where $u \in C^\infty(C)$. Obviously $\varrho_C = \iota^* \varrho_M \operatorname{prol}$.

Now let us describe the quantised version. Firstly we consider the case of a strongly invariant star product. We take the following star product (anti-standard ordered in the first variables and standard ordered in the second variables) on M

$$f * g = \mu \circ \mathrm{e}^{\frac{\lambda}{\mathrm{i}}\left(\frac{\partial}{\partial\varphi}\otimes\frac{\partial}{\partial p} + \frac{\partial}{\partial J}\otimes\frac{\partial}{\partial\psi}\right)} f \otimes g \tag{86}$$

(where $\mu(f \otimes g) = fg$ denotes the pointwise multiplication) which is indeed an associative deformation of μ with the correct Poisson bracket in first order. Now the quantised Koszul operator $\boldsymbol{\partial}_1$ is given by right multiplication with J. Due to the particular form of $*$ it is simply the undeformed multiplication

$$\boldsymbol{\partial}_1 f = f * J = fJ = \partial_1 f. \tag{87}$$

Thus we do not have to deform the chain homotopy h_0 and the restriction ι^* and simply have $\mathsf{id} = \operatorname{prol}\iota^* + \boldsymbol{\partial}_1 h_0$. Finally we compute the quantum Lie algebra action by taking $*$-commutators with J. It turns out that $*$ is strongly invariant, i.e. we have

$$\boldsymbol{\varrho}_M f = \frac{1}{\mathrm{i}\lambda}\operatorname{ad}_*(J) f = \frac{1}{\mathrm{i}\lambda}(J * f - f * J) = \{J, f\} = -\frac{\partial}{\partial\psi} f = \varrho_M f, \tag{88}$$

and similarly $\boldsymbol{\varrho}_C = \iota^* \boldsymbol{\varrho}_M \operatorname{prol} = \varrho_C$ for the quantum Lie algebra action on C. Thus the invariant functions on C are clearly in bijection to the functions on M_{red} via π^* and hence this Hamiltonian quantum $\mathfrak{u}(1)$-space allows a consistent quantum reduction by setting $\boldsymbol{E} = \boldsymbol{E}_0 = E$. The star product $*_{\mathrm{red}}$ on the reduced phase space is well-defined by $\pi^*(u *_{\mathrm{red}} v) = \iota^*((\operatorname{prol}\pi^* u) * (\operatorname{prol}\pi^* v))$, which explicitly yields the expected result

$$u *_{\mathrm{red}} v = \mu \circ \mathrm{e}^{\frac{\lambda}{\mathrm{i}}\frac{\partial}{\partial\varphi}\otimes\frac{\partial}{\partial p}} u \otimes v. \tag{89}$$

Next we shall consider what happens if one does not use a strongly invariant star product. Note that for a one-dimensional Lie algebra any star product is necessarily covariant. We shall even use a star product $\tilde{*}$ equivalent to $*$. Consider the equivalence transformation

$$S = \exp\left(\lambda P \frac{\partial}{\partial J}\right) \tag{90}$$

where $P \in C^\infty(M)$ is a smooth function of the momentum p alone. Then define

$$f \,\tilde{*}\, g = S\left(S^{-1} f * S^{-1} g\right), \tag{91}$$

and obtain $SJ = J - \lambda P$ since $\frac{\partial}{\partial J} P = 0$. A straightforward computation yields that this star product is in general no longer strongly invariant but

$$\tilde{\boldsymbol{\varrho}}_M f = \frac{1}{\mathrm{i}\lambda}(J \,\tilde{*}\, f - f \,\tilde{*}\, J) = \{J, f\} + \mathrm{i}\, \mathrm{ad}_*(P) f, \tag{92}$$

where ad_* denotes the commutator with respect to $*$. Here one uses the fact that S commutes with differentiation in the φ- and ψ-direction. Note that $\mathrm{ad}_*(P)$ differentiates only in the φ-direction and has coefficients depending on p only. The quantum Koszul operator for this star product can be computed also explicitly yielding

$$\tilde{\boldsymbol{\partial}}_1 f = f \,\tilde{*}\, J = \partial f + \lambda f P - \lambda f * P, \tag{93}$$

where we used the fact that S is an automorphism of the undeformed product and acts trivially on functions which do not depend on J. The operator B_1 is then given by

$$B_1 f = f * P - f P = -\mathrm{ad}_*(P) f. \tag{94}$$

Let us now compute the quantum restriction $\boldsymbol{\iota}^* = \iota^*(\mathsf{id} - \lambda B_1 h_0)^{-1}$ more explicitly. First notice that B_1 commutes with h_0. Then a simple computation using the Taylor expansion in J around $J = 0$ yields

$$\iota^* \, h_0^k = \frac{1}{k!} \, \iota^* \, \frac{\partial^k}{\partial J^k} \tag{95}$$

for all $k \in \mathbb{N}$. Together with the fact that $\frac{\partial}{\partial J}$ commutes with $B_1 = -\mathrm{ad}_*(P)$, too, we finally obtain

$$\boldsymbol{\iota}^* = \iota^* \exp\left(-\lambda \mathrm{ad}_*(P) \frac{\partial}{\partial J}\right), \tag{96}$$

illustrating Lemma 27. This enables us to compute the quantum Lie algebra action on C explicitly

$$\tilde{\boldsymbol{\varrho}}_C u = -\frac{\partial u}{\partial \psi} + \mathrm{i}\, \mathrm{ad}_*(P) u, \tag{97}$$

where we can use the operator $\mathrm{ad}_*(P)$ also for functions on C since it only contains differentiation in the φ-direction and multiplication by functions depending on p due to our particular choice of P. Now let us use a more particular function for P, namely we take $P(z, p, w, J) = p$ then $\mathrm{ad}_*(P) = \mathrm{i}\lambda \frac{\partial}{\partial \varphi}$. Now we consider the functions on the constraint surface C which are invariant under this quantum action $\tilde{\boldsymbol{\varrho}}_C$. Let $u = \sum_{r=0}^{\infty} \lambda^r u_r \in C^\infty(C)[[\lambda]]$ then $\tilde{\boldsymbol{\varrho}}_C u = 0$ implies in lowest order $\frac{\partial u_0}{\partial \psi} = 0$ as classically expected. Thus u_0 does not depend on ψ, i.e. is constant along the second circle in $C = T^* S^1 \times S^1$. The next order then implies

$$\frac{\partial u_1}{\partial \psi} = -\frac{\partial u_0}{\partial \varphi} \quad \text{resp.} \quad \frac{\partial u_r}{\partial \psi} = -\frac{\partial u_{r-1}}{\partial \varphi} \tag{98}$$

for $r \geq 1$. Since S^1 is compact we can integrate this equation over the second circle. Together with the fact that u_0 does not depend on ψ this yields that $\frac{\partial u_0}{\partial \varphi}$ has to vanish too. By induction we conclude that $\tilde{\boldsymbol{\varrho}}_C u = 0$ if and only if u does only depend on p but not on φ. Thus we do *not* obtain all functions on the reduced phase space in this case but only those which are constant along the φ-direction. Thus we conclude that in this example there is no consistent quantum reduction.

8 Proper group actions and other nice cases

In view of the examples given in the preceding section we should like to present some classes of Hamiltonian quantum $\mathfrak{g}$-spaces $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ with regular constraint surface which allow a consistent quantum reduction. Moreover we shall specify how the star product for the reduced phase space M_{red} looks whenever M_{red} exists as manifold.

The first class of examples is the class of *proper Hamiltonian G-spaces and strongly invariant star products*:

Theorem 32 *Let (M, ω, G, J, C) be a Hamiltonian G-space with regular constraint surface where the Lie group G is connected and acts properly on M. Moreover, pick any strongly invariant star product $*$ on M (whose existence is assured by Fedosov's Theorem), and set $\boldsymbol{J} := J$. Furthermore choose G-equivariant chain homotopies h and a G-equivariant prolongation* prol *(whose existence is assured by Lemma 3). Then we have the following:*

i.) The quantum representations $\boldsymbol{\varrho}_M$ and $\boldsymbol{\varrho}_C$ of the Lie algebra $\mathfrak{g}$ of G are equal to the corresponding classical representations ϱ_M and ϱ_C which implies that the quantum and classical Chevalley-Eilenberg differentials are equal, i.e. $\boldsymbol{\delta} = \delta$ and $\boldsymbol{\delta}^c = \delta^c$. Hence

$$\boldsymbol{H}^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)[[\lambda]]) = H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C))[[\lambda]],$$

whence $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ admits a consistent quantum reduction. Moreover, we have $\hat{\boldsymbol{\delta}}\hat{\boldsymbol{h}} + \hat{\boldsymbol{h}}\hat{\boldsymbol{\delta}} = 0$.

ii.) The star product on the Chevalley-Eilenberg cohomology $H_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C))[[\lambda]]$ is given by the following simplified formula for $c_1, c_2 \in \bigwedge \mathfrak{g}^* \otimes C^\infty(C)[[\lambda]]$ with $\delta c_1 = 0 = \delta c_2$:

$$[c_1] * [c_2] = [\boldsymbol{\iota}^*((\mathrm{prol}\, c_1) \star_{\mathrm{S}} (\mathrm{prol}\, c_2))] \,.$$

iii.) Suppose in addition that the G-action on C is free, and let $\pi : C \to M_{\mathrm{red}} := C/G$ the canonical projection onto the reduced phase space M_{red}. Then there is the following simple formula for the reduced star product $*_{\mathrm{red}}$ of two functions $\phi_1, \phi_2 \in C^\infty(M_{\mathrm{red}})[[\lambda]]$ which is induced by the above construction:

$$\pi^*(\phi_1 *_{\mathrm{red}} \phi_2) = \boldsymbol{\iota}^*((\mathrm{prol}\, \pi^*\phi_1) * (\mathrm{prol}\, \pi^*\phi_2)). \tag{99}$$

Suppose that the prolongation and the chain homotopies are geometric then the star product $*_{\mathrm{red}}$ is bidifferential and if $*$ is even of Vey type then $*_{\mathrm{red}}$ is of Vey type, too.

Moreover, for any two G-invariant functions f_1 and f_2 in $C^\infty(M)[[\lambda]]$ we have the following: let $f_{1\mathrm{red}}$ and $f_{2\mathrm{red}}$ be the unique functions of $C^\infty(M_{\mathrm{red}})[[\lambda]]$ such that $\boldsymbol{\iota}^* f_k =: \pi^* f_{k\mathrm{red}}$ for $k = 1, 2$. Then

$$\pi^*(f_{1\mathrm{red}} *_{\mathrm{red}} f_{2\mathrm{red}}) = \boldsymbol{\iota}^*(f_1 * f_2). \tag{100}$$

iv.) The choice of a different G-invariant geometric prolongation and different geometric chain homotopies yields in general a different but equivalent reduced star product.

Proof: The first part is clear: one has equality of quantum and classical representations since $*$ is strongly invariant, since prol intertwines the classical $\mathfrak{g}$-actions on M and on C in formula (70), and since $\boldsymbol{\iota}^*$ prol is the identity on $C^\infty(C)[[\lambda]]$ by (63). Moreover $\hat{\boldsymbol{\delta}}$ anticommutes with $\hat{\boldsymbol{h}}$ because the quantized Koszul operator, the quantized restriction and the quantized chain homotopies are clearly G-equivariant since their

classical counterparts are G-equivariant. The second part is an immediate consequence of formula (78) and the fact that $\hat{\boldsymbol{h}}' = \frac{1}{2}\boldsymbol{h}$ by the above. The formulas in the third part are simple consequences of part ii.) and of the fact that $H^0_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)) = \pi^* C^\infty(M_{\mathrm{red}})$, see the Propositions 4 and 7. In order to prove the differentiability properties of the reduced star product we consider around some arbitrarily chosen point $x \in M_{\mathrm{red}}$ an open chart U such that the principal fibre bundle C trivialises, hence $\pi^{-1}(U)$ is G-equivariantly diffeomorphic to $U \times G$. Taking the G-equivariant tubular neighbourhood of C restricted to $\pi^{-1}(U)$ we finally get a chart domain of M diffeomorphic to an open subset of $U \times G \times \mathfrak{g}^*$. In both cases above the star product restricted to this domain is a series of bidifferential operators. Since $\operatorname{prol} \pi^* \phi_1$ simply corresponds to a function on this domain not depending on the coordinates on $G \times \mathfrak{g}^*$ and since the quantum restriction map $\boldsymbol{\iota^*}$ is equal to $\iota^* \circ S$ where S is a series of differential operators, or differential operators of order bounded by the order of λ, respectively, according to Lemma 27, we immediately see in co-ordinates that the resulting bilinear operators of the reduced star products are bidifferential operators of the same order which proves the two asserted properties at the same time. The last formula follows from the fact that $f - \operatorname{prol} \pi^* f_{\mathrm{red}}$ is in the intersection of the quantum vanishing ideal with the space of all smooth complex-valued G-invariant functions, $C^\infty(M)^G[[\lambda]]$: this intersection is a two-sided ideal of the subalgebra $C^\infty(M)^G[[\lambda]]$ which follows from the explicit form of the elements of $\boldsymbol{\mathcal{I}}_C$ (see Proposition 23) and the fact that $\langle J, \xi\rangle * f = f * \langle J, \xi\rangle$ for all $\xi \in \mathfrak{g}$ and all $f \in C^\infty(M)^G[[\lambda]]$ since $*$ is strongly invariant. Fourthly, let prol' and $\boldsymbol{\iota^*}'$ be another choice and let S, S' be G-invariant differential operators on M according to Lemma 27 such that $\boldsymbol{\iota^*} = \iota^* \circ S$ and $\boldsymbol{\iota^*}' = \iota^* \circ S'$. Then the linear map $\Phi : C^\infty(M_{\mathrm{red}})[[\lambda]] \to C^\infty(M_{\mathrm{red}})[[\lambda]]$ defined by $\Phi(\varphi) := (S^{-1} S' \operatorname{prol} \pi^* \varphi)_{\mathrm{red}}$ is easily seen to be an algebra isomorphism of $(C^\infty(M_{\mathrm{red}})[[\lambda]], *_{\mathrm{red}})$ onto $(C^\infty(M_{\mathrm{red}})[[\lambda]], *'_{\mathrm{red}})$ using the fact that $(\cdot)_{\mathrm{red}}$ is a homomorphism and that $f_{\mathrm{red}'} = (S^{-1} S' f)_{\mathrm{red}} = \Phi f_{\mathrm{red}}$ for each $f \in C^\infty(M)^G[[\lambda]]$. The fact that Φ is a formal series of differential operators is shown by a local consideration analogous to the one in part three. $\square$

Remark 33 *The associativity of the explicit formula (99) can also be seen more directly: note that the space of G-invariant functions $C^\infty(M)^G[[\lambda]]$ on M, is a sub-algebra of $(C^\infty(M)[[\lambda]], *)$ due to the G-invariance of $*$. Furthermore, since the projection $\operatorname{prol} \boldsymbol{\iota^*}$ is G-equivariant we have the decomposition*

$$C^\infty(M)^G[[\lambda]] = (C^\infty(M)^G[[\lambda]] \cap \mathcal{F}_C[[\lambda]]) \oplus (C^\infty(M)^G[[\lambda]] \cap \boldsymbol{\mathcal{I}}_C).$$

As has been shown above the space $C^\infty(M)^G[[\lambda]] \cap \boldsymbol{\mathcal{I}}_C$ is a two-sided ideal in $C^\infty(M)^G[[\lambda]]$. Hence the $\mathcal{F}_C$-component of the star product of two G-invariant functions yields an associative multiplication. We use results on Koszul homology to prove the directness of the above decomposition. This point of view has been used by Schirmer to compute a star product on complex Graßmann manifolds, see [43, 44].

The second class of examples is based on the following

Theorem 34 *Let $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ be a Hamiltonian quantum $\mathfrak{g}$-space with regular constraint surface. Suppose in addition that the first classical Chevalley-Eilenberg cohomology group on the constraint surface, $H^1_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C))$, vanishes. Then the Hamiltonian quantum $\mathfrak{g}$-space allows a consistent quantum reduction.*

Proof: Write $\boldsymbol{\delta}^c = \sum_{r=0}^\infty \lambda^r \boldsymbol{\delta}^c_r$ where $\boldsymbol{\delta}^c_r$ are linear endomorphisms of $\bigwedge g^* \otimes C^\infty(C)$ and $\boldsymbol{\delta}^c_0 = \delta^c$. We consider the equation

$$\boldsymbol{\delta}^c \phi' = 0$$

where $\phi' = \sum_{r=0}^\infty \lambda^r \phi_r$ is in $C^\infty(C)[[\lambda]]$. Its solvability is a standard argument for deformed cohomology operators (as for instance in the proof of the existence of the Fedosov construction): Choose a vector subspace

$\mathcal{C} \subset C^\infty(C)$ such that $C^\infty(C) = C^\infty(C)^{\mathfrak{g}} \oplus \mathcal{C}$. We construct the maps $E_r, r \in \mathbb{N} : C^\infty(C)^{\mathfrak{g}} \to C^\infty(C)$ as follows: let $\phi \in C^\infty(C)^{\mathfrak{g}}$. Hence $\delta^c\phi = 0$ and the above equation is satisfied up to order zero. Set $\boldsymbol{E}_0$ equal to the canonical injection into $C^\infty(C)$ and suppose that the maps $\boldsymbol{E}_r$ have already been constructed up to order $s \in \mathbb{N}$ such that the image of $\boldsymbol{E}_r$ is contained in the complementary space $\mathcal{C}$ for all $1 \leq r \leq s$, and such that $\phi^{(s)} := \sum_{r=0}^{s} \lambda^r \boldsymbol{E}_r(\phi)$ solves the above equation up to order s, i.e. $(\boldsymbol{\delta}^c\phi^{(s)})_r = 0$ for all $0 \leq r \leq s$. Consider $(\boldsymbol{\delta}^c\phi^{(s)})_{s+1}$. Since obviously $\boldsymbol{\delta}^c(\boldsymbol{\delta}^c\phi^{(s)}) = 0$ we get $0 = (\boldsymbol{\delta}^c(\boldsymbol{\delta}^c\phi^{(s)}))_{s+1} = \delta^c(\boldsymbol{\delta}^c\phi^{(s)})_{s+1}$. But since $H^1_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C))$ vanishes there must be a $\phi_{s+1} \in C^\infty(C)$ such that

$$(\boldsymbol{\delta}^c\phi^{(s)})_{s+1} = -\delta^c\phi_{s+1}.$$

Clearly ϕ_{s+1} is unique up to addition of an arbitrary element $\psi_{s+1} \in H^0_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C))$. Let $\boldsymbol{E}_{s+1}(\phi)$ be the unique such element ϕ_{s+1} in $\mathcal{C}$. This clearly defines a linear map $\boldsymbol{E}_{s+1} : C^\infty(C)^{\mathfrak{g}} \to \mathcal{C} \subset C^\infty(C)$, and the above equation is clearly equivalent to the equation $0 = (\boldsymbol{\delta}^c\phi^{(s+1)})_{s+1}$. Proceeding this way by induction we construct all the maps $\boldsymbol{E}_r$ such that $\phi' = \boldsymbol{E}\phi = \sum_{r=0}^{\infty} \lambda^r \boldsymbol{E}_r(\phi)$ solves $\boldsymbol{\delta}^c\phi' = 0$ for arbitrary $\phi \in C^\infty(C)^{\mathfrak{g}}$. Since on the other hand every solution to $\boldsymbol{\delta}^c\phi' = 0$ can obviously constructed that way the theorem is proved. $\square$

The cohomological criterion in the preceding theorem can be made more explicit in case the Hamiltonian quantum $\mathfrak{g}$-space comes from a Hamiltonian G-space with existing reduced phase space:

Lemma 35 *Let (M, ω, G, J, C) be a Hamiltonian G-space with regular constraint surface where the connected Lie group freely and properly acts on C. Then $H^1_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)) = 0$ if and only if the first de Rham cohomology group of the Lie group G vanishes.*

PROOF: We know that C is a principal fibre bundle over the reduced phase space $M_{\mathrm{red}} := C/G$ with structure group G. Observe that $H^\bullet_{\mathrm{de\ Rham}}(G) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g}, C^\infty(G))$ using right invariant differential forms as a global basis of all differential forms on G and letting $\mathfrak{g}$ act on $C^\infty(G)$ by Lie derivatives of right invariant vector fields. Moreover δ^c clearly commutes with multiplication by pull-backs of smooth functions on the reduced phase space by means of the projection π. Hence, using partitions of unity on the reduced space it suffices to prove the assertion for cochains supported in bundle charts diffeomorphic to $U \times G$. If $H^1_{\mathrm{CE}}(\mathfrak{g}, C^\infty(C)) = 0$ choose a closed one-form β on G, i.e. an element in $\mathfrak{g}^* \otimes C^\infty(G)$, multiply it with a bump function b being equal to 1 at some point x on the reduced space to obtain a δ^c-closed element β' in $\mathfrak{g}^* \otimes C^\infty(C)$. Since $\beta' = \delta^c\phi$ then $\beta = d\phi(x, \cdot)$, showing one implication. For the reverse implication note that every $\beta' \in \mathfrak{g}^* \otimes C^\infty(C)$ which is δ^c-closed gives rise to a d-closed one-form $\beta'(u, \cdot)$ on G smoothly parametrised by $u \in U$. For instance by means of line integrals on G along arbitrary smooth paths starting at the neutral element we can choose a smooth function $\phi \in C^\infty(U \times G)$ such that $\beta'(u, \cdot) = d\phi(u, \cdot)$. Hence β' is equal to $\delta^c\phi$, and this will establish the reverse implication. $\square$

9 Outlook and open problems

In this section we list some open questions arising with our approach to BRST cohomology and give an outlook to future work we plan to do.

The first remark concerns the principal question of covariant star products. Obviously our approach relies crucially on the *existence of an at least quantum covariant star product* for the given Lie group or Lie algebra action. Though in the literature many examples of the existence of (quantum) covariant star products are known, it seems that there is no general theorem on the existence of covariant star products. Neither it is known whether there are principal obstructions for the construction of covariant star products. Thus it would be highly desirable to find here more concrete conditions whether a covariant star product exists or even prove that there are no obstructions at all beside the existence of a classical momentum map. An answer to this

question would have many useful consequences not only for the BRST method. In particular the general existence of a covariant star product would imply the *quantum integrability* of any classically integrable system where one simply has a Hamiltonian $\mathbb{R}^n$-action.

Secondly, in view of the counter example in Section 7, there is the following open question: For any Hamiltonian quantum $\mathfrak{g}$-space with regular constraint surface $(M, *, \mathfrak{g}, \boldsymbol{J}, C)$ does there always exist an equivalent star product $*'$ on M such that $(M, *', \mathfrak{g}, \boldsymbol{J}, C)$ is a Hamiltonian quantum $\mathfrak{g}$-space with regular constraint surface allowing for a consistent quantum reduction? Moreover, it may happen that equivalent but different star products on M induce non-equivalent star products on M_{red}. Such effects are known from phase space reduction of star products in several examples, see e.g. [24, 25, 46], and thus have to be expected in the BRST framework as well. Finally it would be interesting to see how one can compute the Deligne class (see [14] and [27] for definitions) of the reduced star product in terms of the BRST algebra. As a conversation of M.B. with G. Halbout and, independently, a proposal of the referee suggests this will probably be related to a formulation of the (BRST) reduction in terms of a suitable 'equivariant cohomology'.

As third open problem we would like to mention that the formalism we have presented here only deals with the *observable algebra*. Though the observables are the primary object of deformation quantization one has to deal with the question how the *states* of the considered physical system can be described. In [11] the concept of formally positive functionals and their GNS representations in deformation quantization has been introduced and shown to be a physically reasonable concept for the description of states. Thus we plan in a forthcoming paper to consider GNS representations of the BRST algebra. Here some subtleties arise from the fact that the *-involution of complex conjugation is now $\mathbb{Z}_2$-graded where we have to use the Weyl ordered case. Thus the concept of positive functionals and their GNS representation has to be modified. The main problem is to find conditions for $\mathbb{C}[[\lambda]]$-linear functionals of the BRST algebra such that firstly the GNS representation of the BRST algebra induces indeed a representation of the algebra of smooth functions on the reduced phase space, and, secondly, to guarantee the positivity of the inner product of the representation space of the reduced observables. The problem of a positive inner product is known in the usual approach to BRST quantization and the framework of formal GNS representations seems to be a promising way to construct positive inner products.

After the first version of this paper appeared on the internet M. Semenov-Tyan-Shanskii proposed to M.B. to investigate the more general problem of a Lie-Poisson group action with a 'noncommutative momentum map' taking values in the dual group: here a (quantum) BRST formulation of the phase space reduction does not yet seem to have been achieved.

Fifthly, apart from BRST theory the more general problem of finding *deformed or quantised analogues of classical sub-manifolds* seems to be interesting and also occurring (see also e.g. [47]): The quantum vanishing ideal $\boldsymbol{\mathcal{I}}_C$ is a left ideal corresponding to the *coisotropic* sub-manifold $C = J^{-1}(\{0\})$. This would also support the 'coisotropic creed' formulated by Lu in [35]. Furthermore, one may speculate whether a connected *Lagrangian* sub-manifold corresponds to a left ideal $\boldsymbol{\mathcal{I}}$ whose Lie idealiser is equal to $\boldsymbol{\mathcal{I}} \oplus \mathbb{C}[[\lambda]]1$. This seems to be reasonable in view of the situation for cotangent bundles: As discussed in detail in [8–10] for any cotangent bundle T^*Q there is a homogeneous star product of Weyl type such that the integration over the configuration space with respect to a positive density is a formally positive functional whose Gel'fand ideal (which is a left ideal) characterises the configuration space as embedded Lagrangian sub-manifold. Here one can rather easily show that the Lie idealiser of this Gel'fand ideal can be obtained by adding the multiples of the function 1 if the configuration space Q is connected. We shall not go into details but mention that the above characterisation is still true for projectable Lagrangian sub-manifolds L if one takes the Gel'fand ideal of a formally positive functional having support on L which also induces the WKB expansion.

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